ENGINEERING GRAPHICS SEMINAR

FOUR-DIMENSIONAL DESCRIPTIVE GEOMETRY

METRIC PROBLEMS: ANGLES - DESCRIPTIVE SOLUTION

C. Ernesto S. Lindgren
Visiting Research Engineer
United States Steel Corporation

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Department of Graphics and Engineering Drawing
School of Engineering and Applied Science
Princeton University

ABSTRACT

The paper presents the descriptive geometry solution of the determination of the angles between geometric elements in a four-dimensional euclidean space.

Incorporated in these solutions are the discussions of some problems of intersections between two geometric elements. These are: intersection of a line and a plane, belonging to the same 3-D space; intersection of two planes belonging to distinct 3-D spaces; intersection of a line and a 3-D space; intersection of a plane and a 3-D space.

Problem No. 1

"To determine the true value of the angle of two lines".

Preliminary discussion: if the two lines are non-concurrent, through a point of one draw a line parallel to the other and determine the angle of the two concurrent lines.

Solution

Consider a 3-D space that belongs to the plane determined by the two concurrent lines, apply a rotation of the 3-D space[1)] until it is superimposed on one of the 3-D spaces of the 4-D system of reference, and use a three-dimensional method of descriptive geometry to determine the true value of the angle.

To determine a 3-D space that belongs to the plane (abc) find the points of intersection of plane (abc) with the planes π_1, π_2, π_3 of the 4-D system of reference. (Figure 2)

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1) See the author's Special Report No. 4, Engineering Graphics Seminar, Department of Graphics and Engineering Drawing, Princeton University, March, 1965.

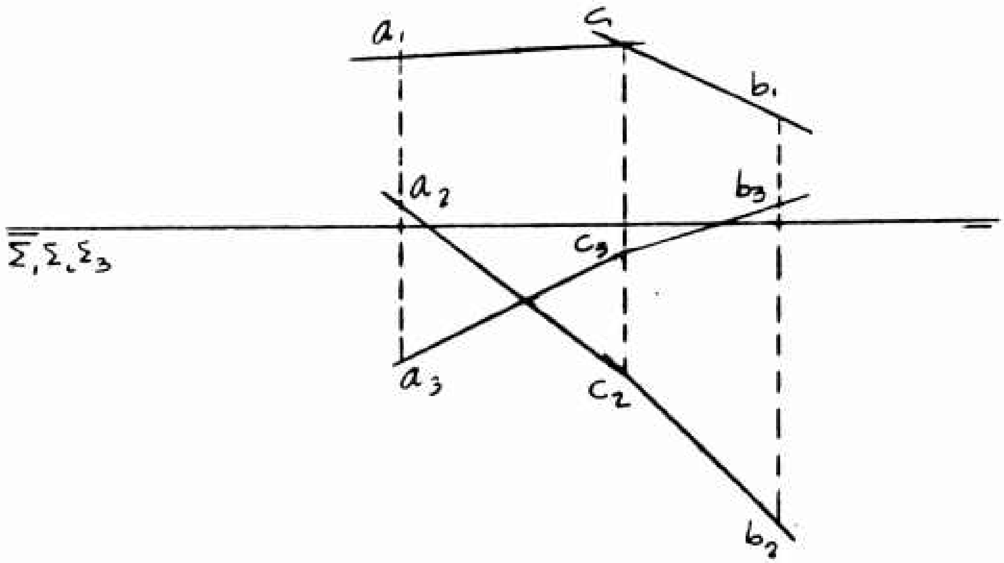

Figure 1

Next, using τ_1, τ_2, τ_3 as axes of reference, determine the position of (a), (b), and (c). See pictorial view in Figure 3 and constructions in the orthographic projection in Figure 4.

To determine the superimposition of 3-D space T on the 3-D space Σ_3 of the 4-D system of reference, consult our Special Report No. 4[2]. The graphical constructions are shown in Figures 5, 6, and 7.

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2) Engineering Graphics Seminar, Technical Seminar Series, Department of Graphics and Engineering Drawing, Princeton University.

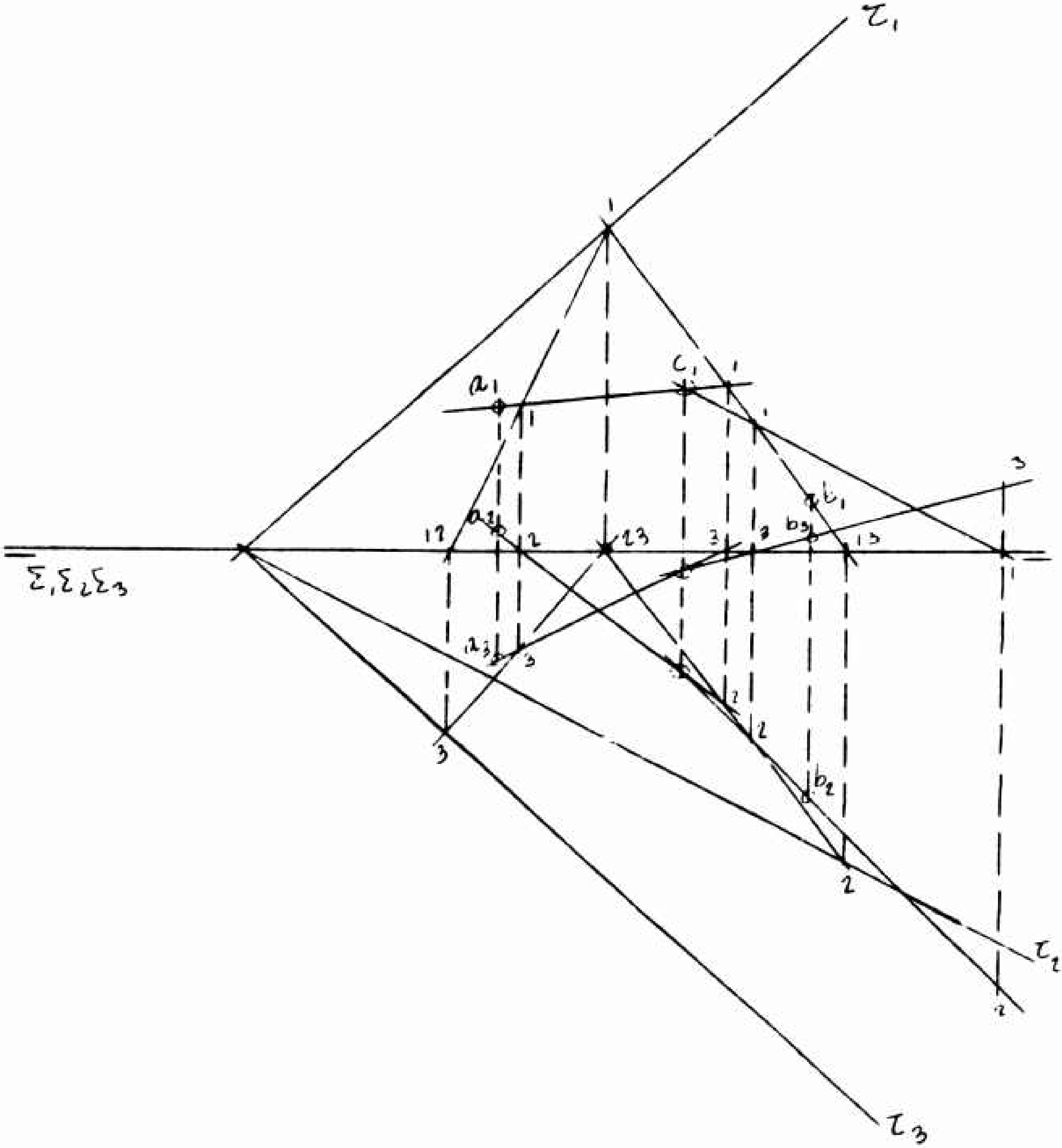


Figure 2

As seen in Figure 7, the line (ob$''$) represents the trace τ_3 of the 3-D space T. We shall proceed by determining the positions of (a$'$) and (c$'$) on that line (Figure 8). The true lengths of (oa$'$) and (oc$'$) are obtained from Figure 5.

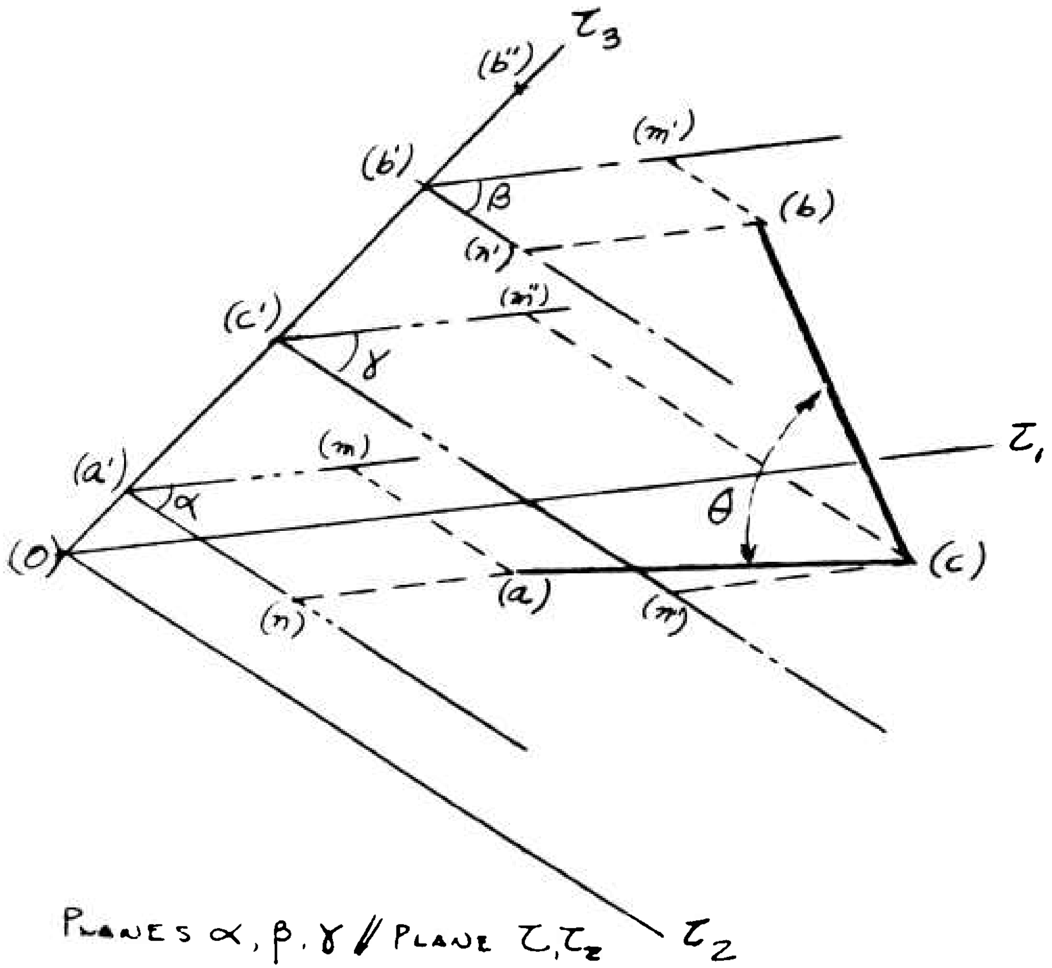


Figure 3

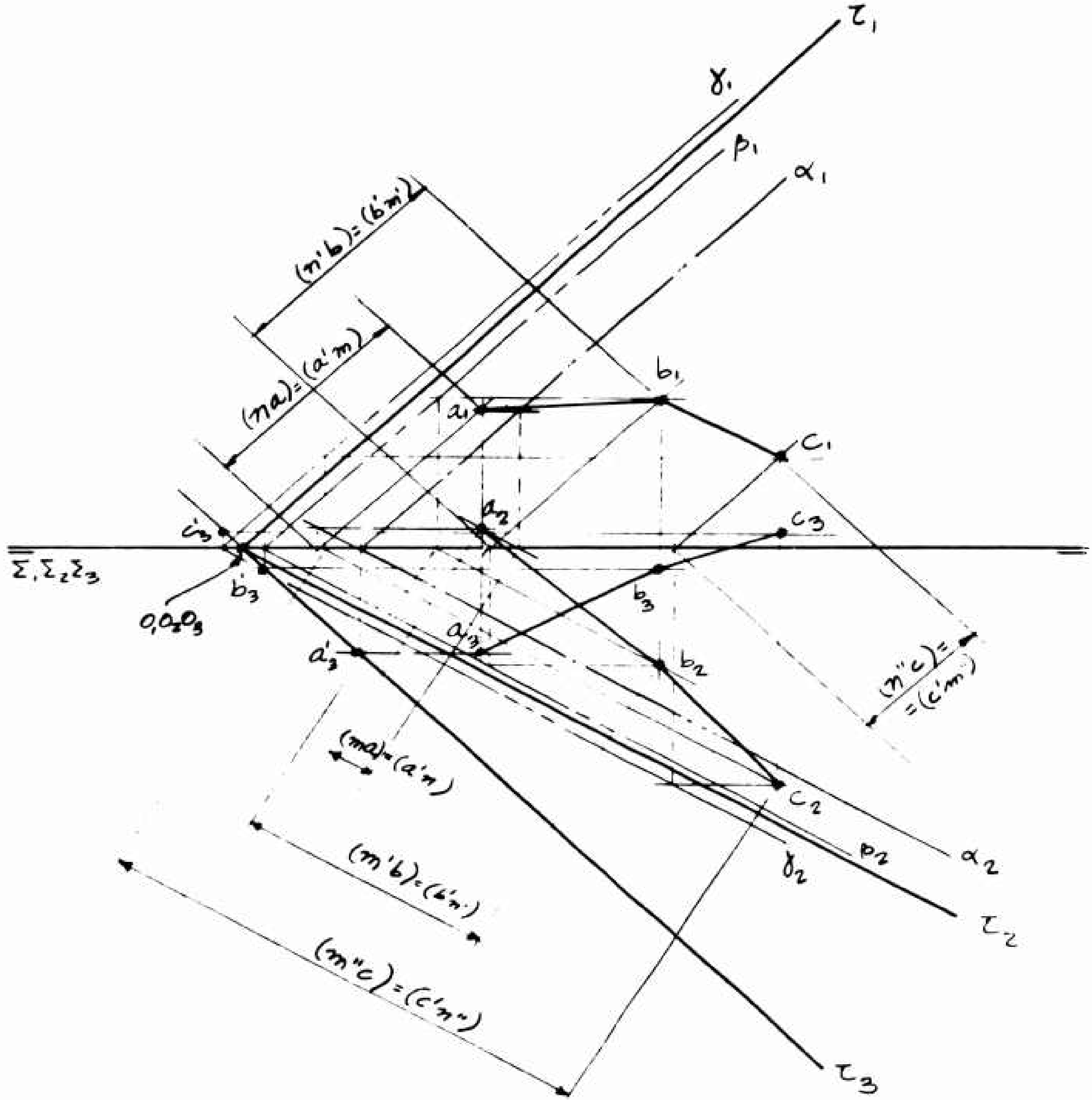

τ₁
γ₁
β₁
α₁
(n'b)=(b'm')
(na)=(a'm)
b₁
a₁
c₁
a₂
c₃
Σ₁Σ₂Σ₃
b₃
O₁O₂O₃
a₃
b₂
(n"c)=
=(c'm")
(ma)=(a'n)
c₂
α₂
β₂
γ₂
τ₂
(m'b)=(b'n')
(m"c)=(c'n")
τ₃

Figure 4

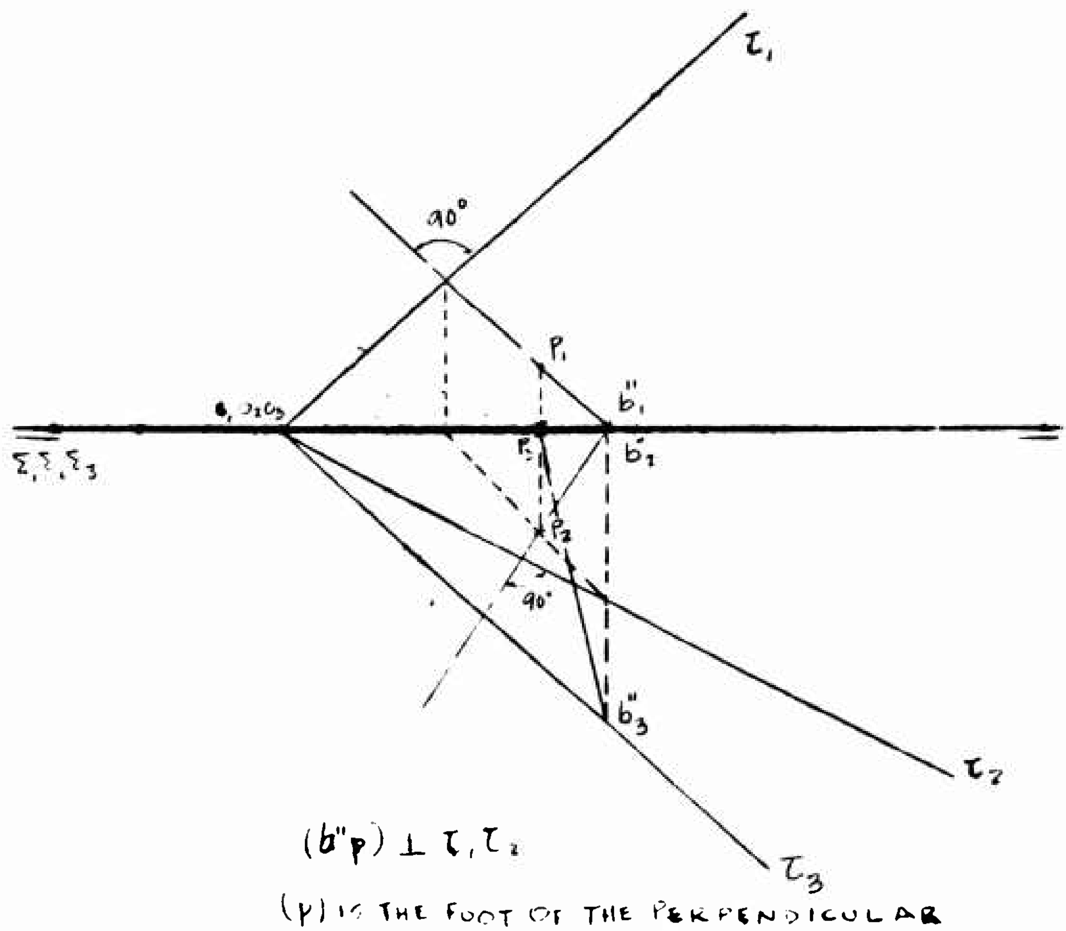


Figure 5

Next, draw ($a'm$), ($b'm'$), ($c'm''$) parallel to τ_1, ($a'n$), ($b'n'$), ($c'n''$) parallel to τ_2, marking its respective lengths obtained from Figure 5. Then, through (m) and (n) draw parallels to τ_2 and τ_1, respectively, to determine (a). Use similar procedures to obtain (b) and (c), working with points (m'), (n'), (m''), and (n''). Figure 9.

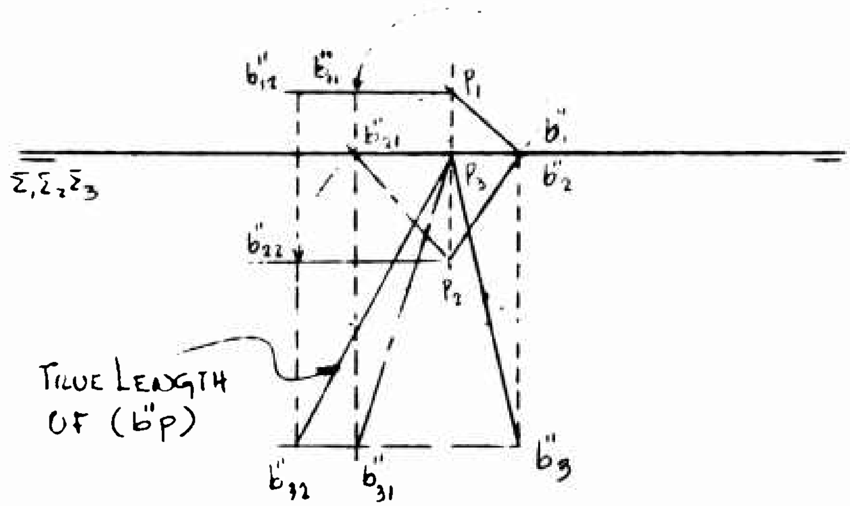


Figure 6

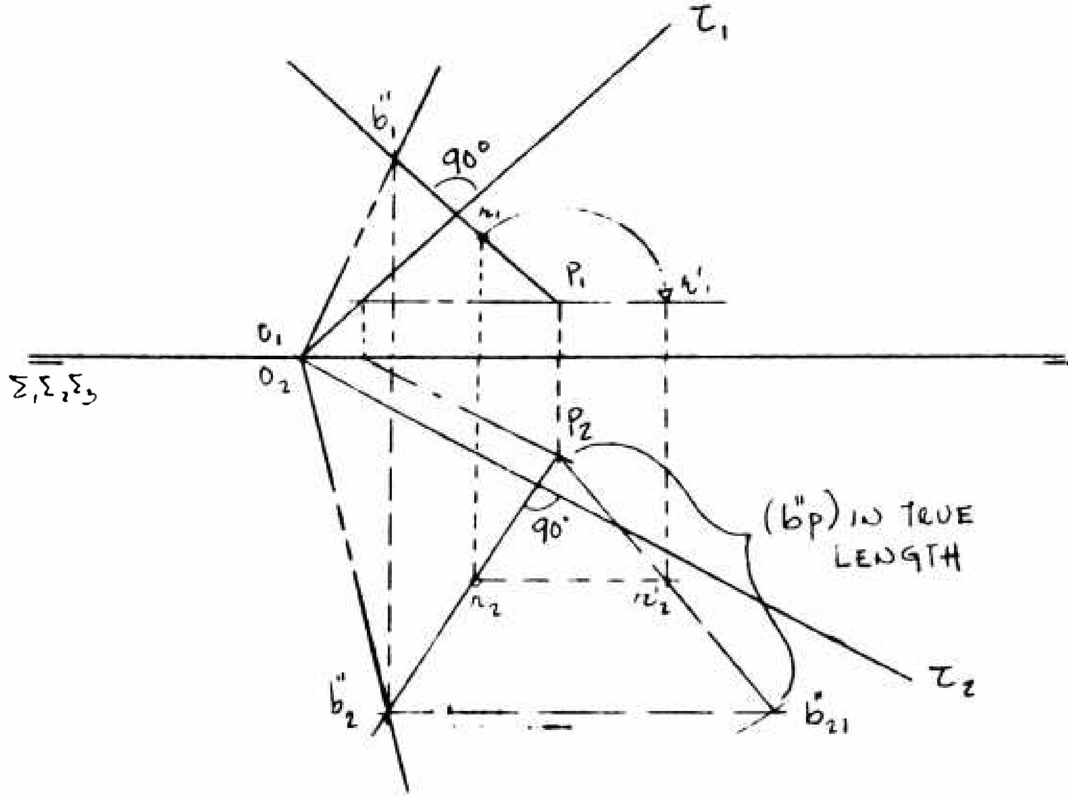


FIGURE 7

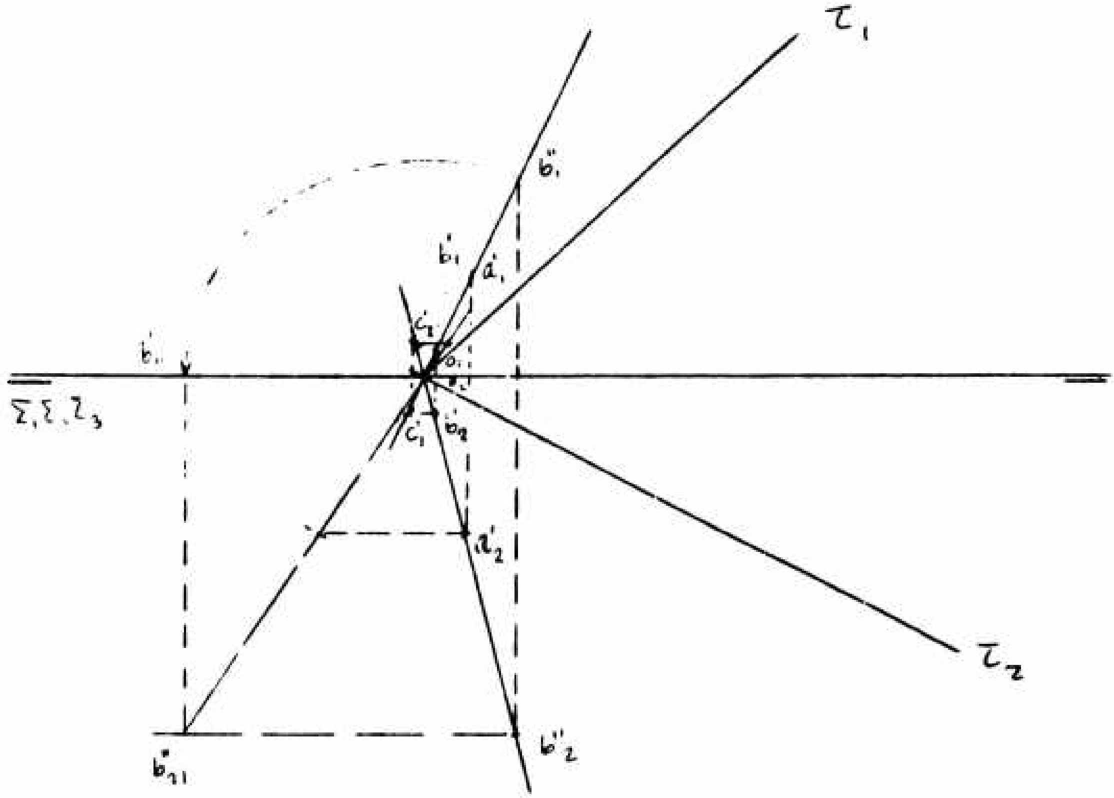


Figure 8

Thus, we obtained the projections of the three points (a), (b), (c), of the 3-D space T, when superimposed on the 3-D space Σ_3. To determine the true value of the angle $(a\hat{c}b)$, use a method of the three-dimensional descriptive geometry. (Figure 10).

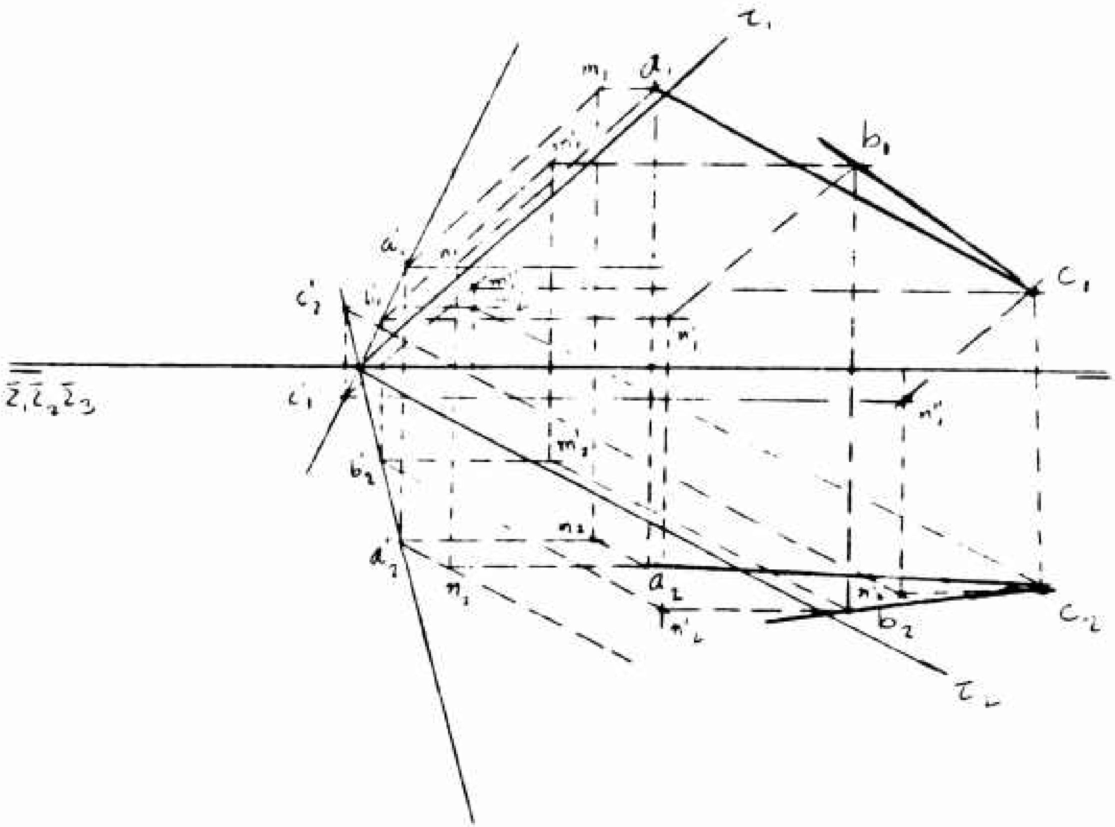


Figure 9

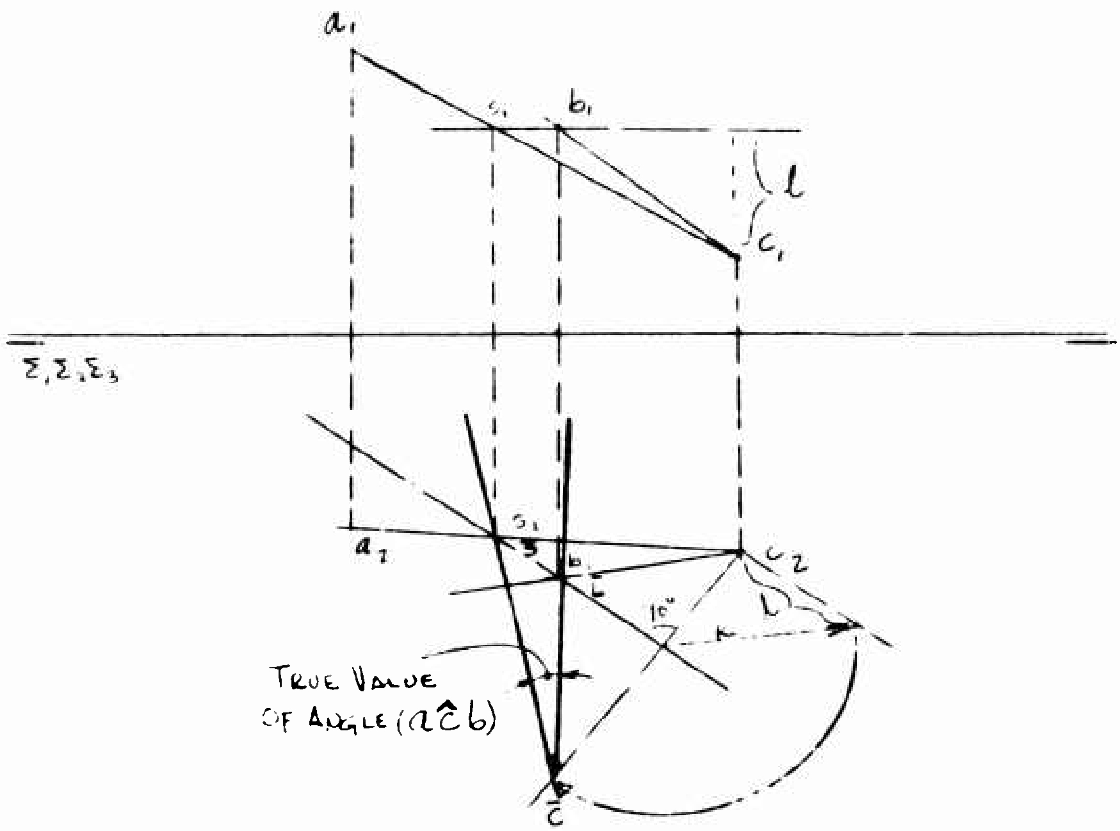


Figure 10

Problem No. 2

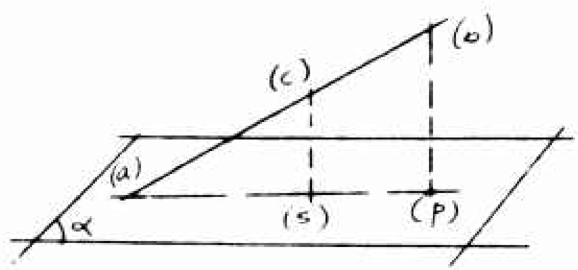
"To determine the true value of the angle of a line and a plane".

Preliminary discussion: if the line and the plane are non-concurrent, (belonging to two distinct 3-D spaces), through a point of the plane draw a line parallel to the given one and determine its angle with the plane.

<u>Solution</u>

The angle of a line with a plane is equal to the angle of the line with its orthogonal projection on the plane.

Thus we should find the point (a), intersection of the line and the plane and through a point (b), of the line draw a perpendicular to the plane and determine its foot, point (p). The angle $(b\hat{a}p)$ is the angle of (ab) with the plane. (Figure 11).



$$\left.\begin{array}{l}(ab) \times \alpha \Rightarrow (a) \\ (bp) \perp \alpha\end{array}\right\} (b\hat{a}p) = \widehat{(ab),\alpha}$$

IF (a) IS INACCESSIBLE

$$\widehat{(cb),(sp)} = \widehat{(ab),\alpha}$$

WHERE $(cs) \perp \alpha$

Figure 11

The general proposition of the problem gives a line (b'c') and a plane α . In Figure 12, the plane α is given by two concurrent lines (R) and (T).

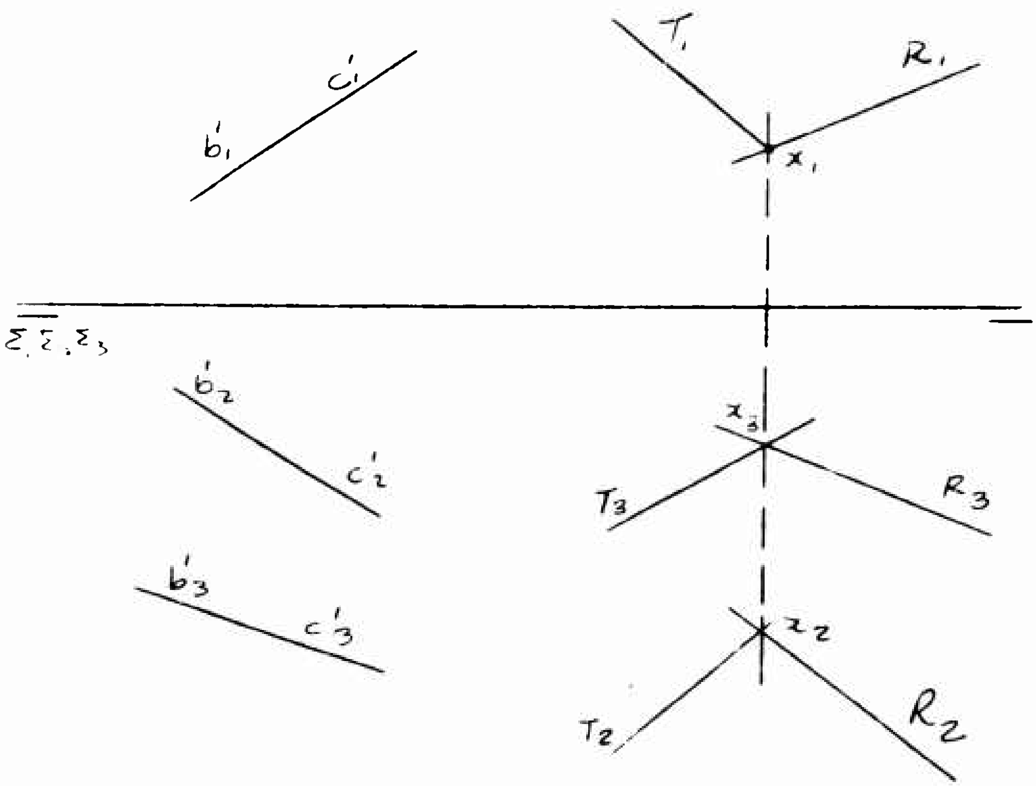


Figure 12

Without investigating if they belong to the same 3-D space, take a point (a) in (R) and draw (ab) parallel to (b'c'). (Figure 13).

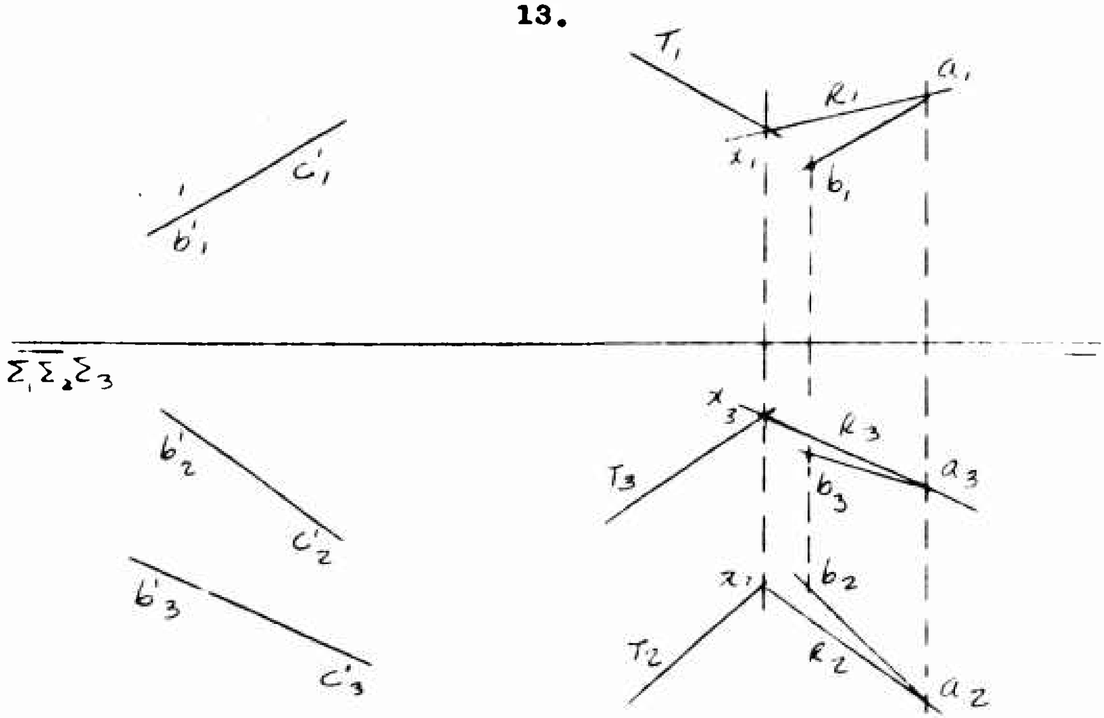
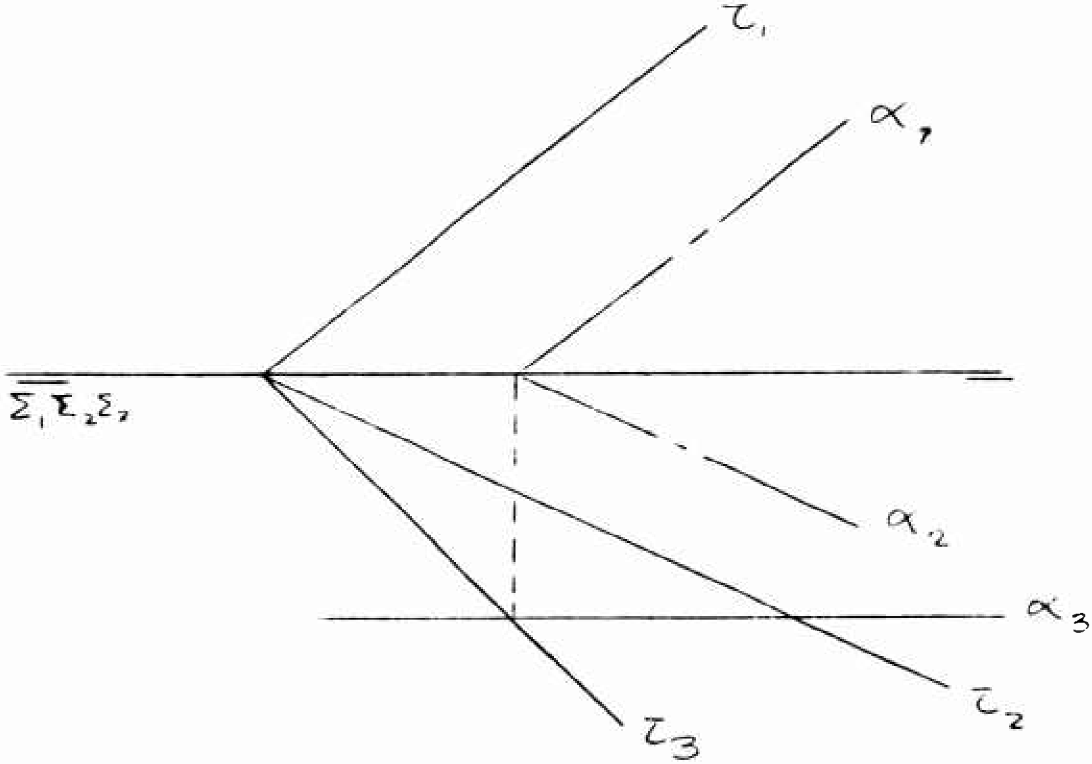


Figure 13

Consider the 3-D space (R, T - b) to proceed with the solution.

Since through (b) we have to draw perpendicular to plane (R, T) $\equiv \alpha$, it will be necessary to apply a descriptive method so that we may properly operate within the 3-D space (R, T - b). The objective should be to place the plane α parallel to one of the 3-D spaces of the 4-D system of reference, as indicated in Figure 14. Here, the plane α is parallel to Σ_3 and belongs to the 3-D space T.



<u>Figure 14</u>

The graphical constructions pertinent to the application of the descriptive method are outlined in other papers[3] and we shall not repeat them here.

Therefore, let us admit the 3-D space $\Pi \equiv (R, T - b)$ of Figure 15, given by its traces, and the plane α and point (b) belonging to it. In the plane α we consider

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3) Descriptive Geometry of Four Dimensions, Engineering Graphics Seminar, Technical Seminar Series, Report No. 9, Department of Graphics and Engineering Drawing, Princeton University, December 5, 1963.

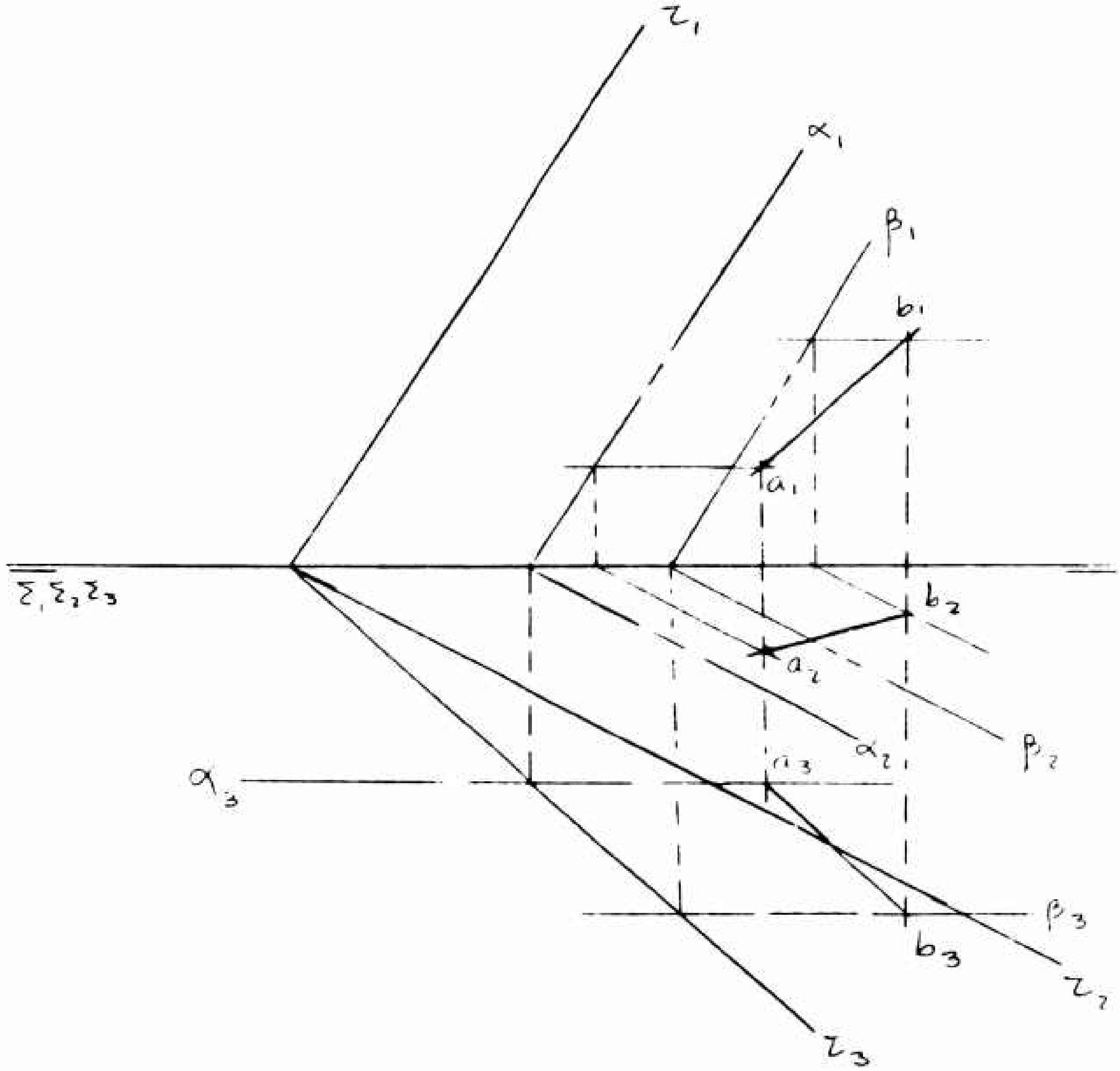


Figure 15

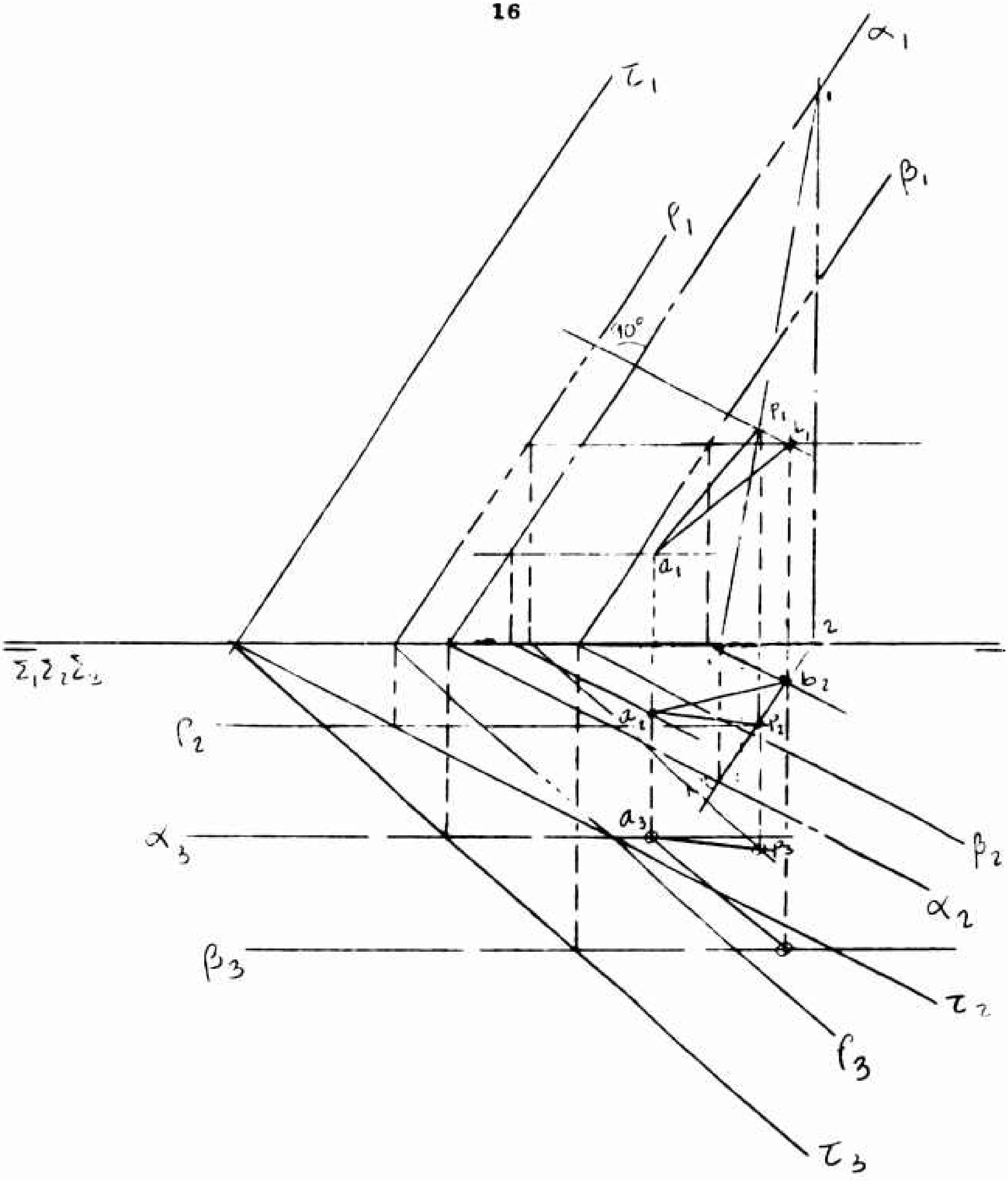

α_1
τ_1
β_1
ρ_1
$10°$
a_1
b_2
a_2
ρ_2
a_3
α_3
β_2
α_2
β_3
τ_2
ρ_3
τ_3

Figure 16

the point (a), and we shall determine the angle that (ab) forms with α.

Through (b) draw the perpendicular to α and determine the foot (p) of the perpendicular. (Figure 16).

To determine the angle of (ab) and (bp) apply the solution outlined for Problem No. 1.

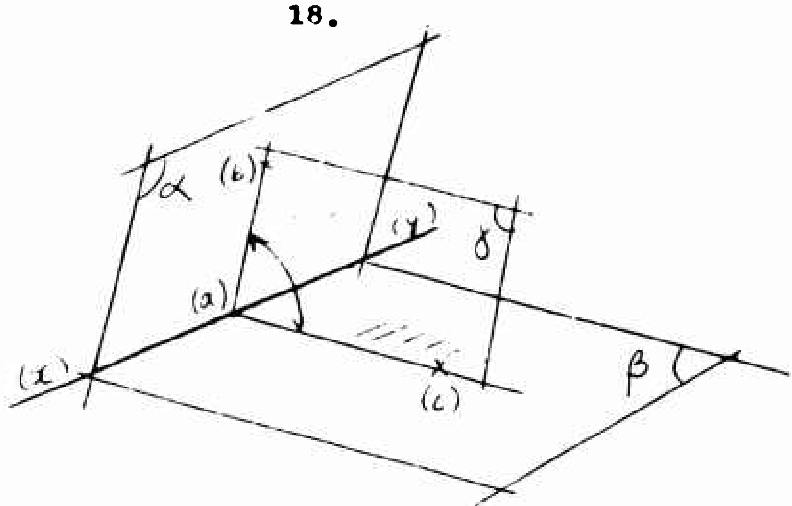
Problem No. 3

"To determine the true value of the angle of two planes that belong to the same 3-D space".

Solution

Let α and β be the given planes, intersecting along a line (xy). Through (a) of (xy) pass a plane γ, perpendicular to (xy) and determine $\alpha \times \gamma \Rightarrow$(ab) and $\beta \times \gamma \Rightarrow$(ac). The true value of the angle ($b\hat{a}c$) is the solution sought. (Figure 17).

In Figure 18, the plane α is given parallel to the 3-D space Σ_3 and the plane β is given by the points (m), (n), (p). The intersection of the two planes is the line (xy).



$\alpha \perp \gamma \; ; \; \beta \perp \gamma \; ; \; \gamma \perp (xy)$

Figure 17

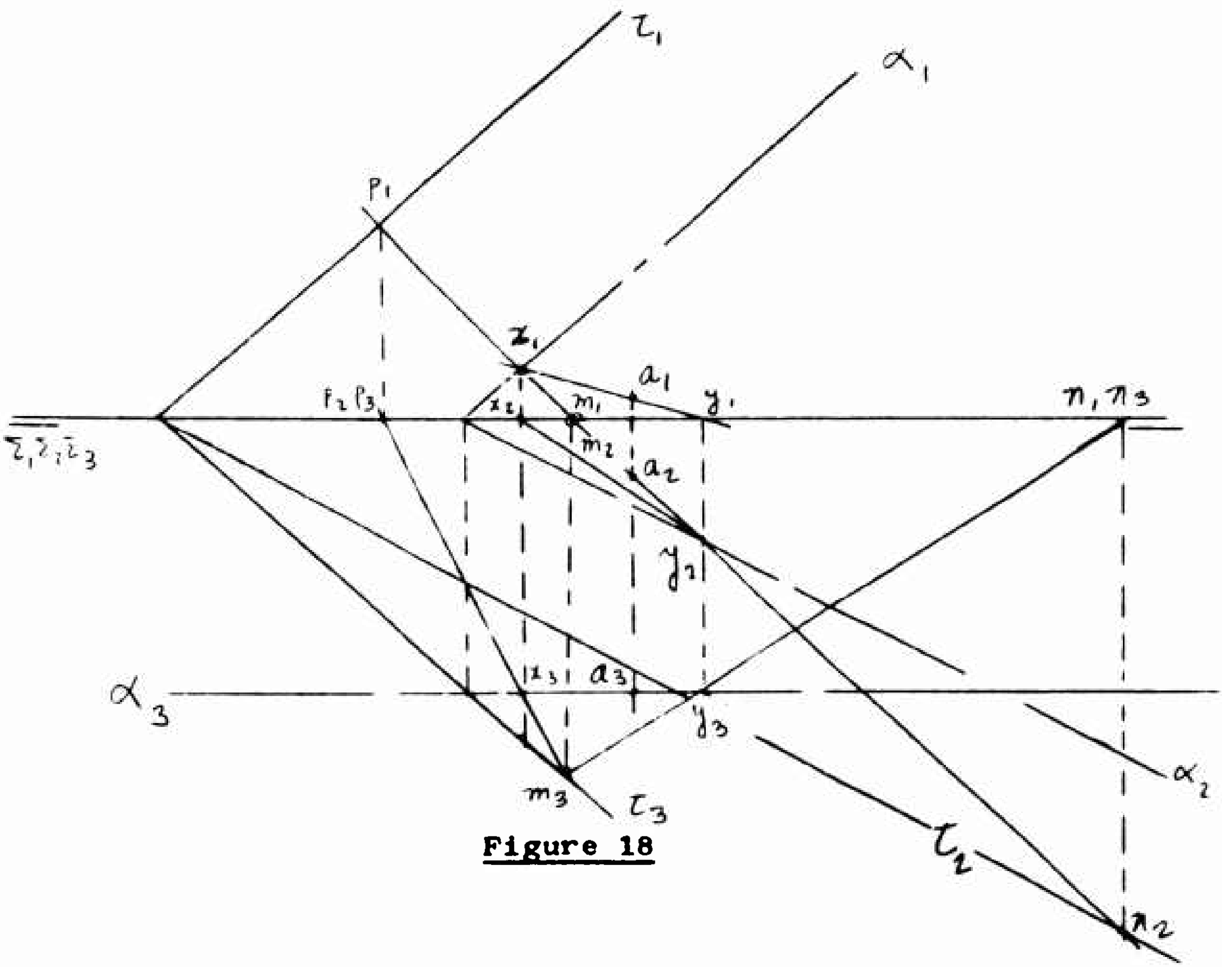


Figure 18

To determine (ab) perpendicular to (xy) in plane α, apply rotation to the plane, about α_2, until superimposition on plane π_2 of the 4-D system of reference. See Figure 19.

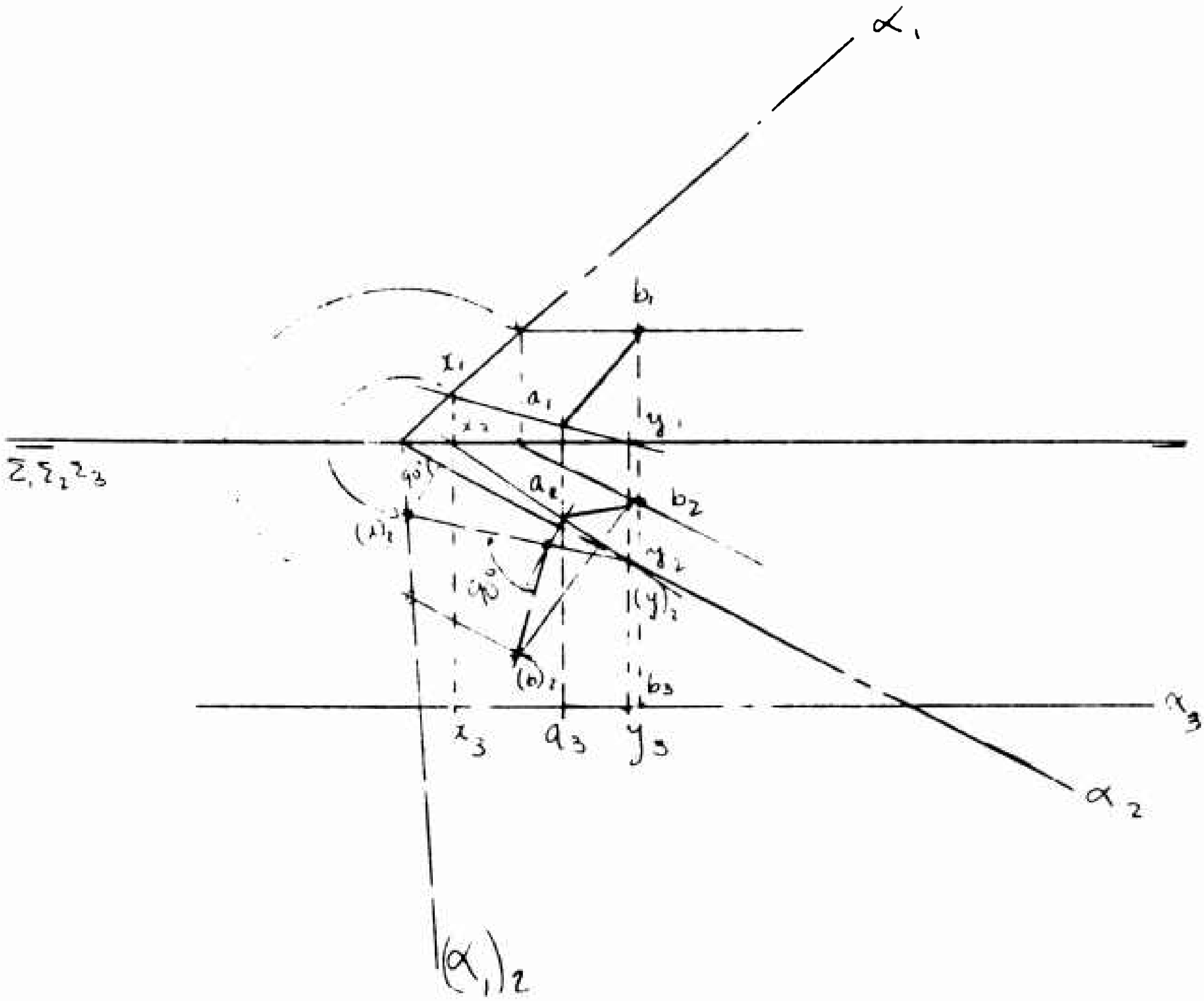


Figure 19

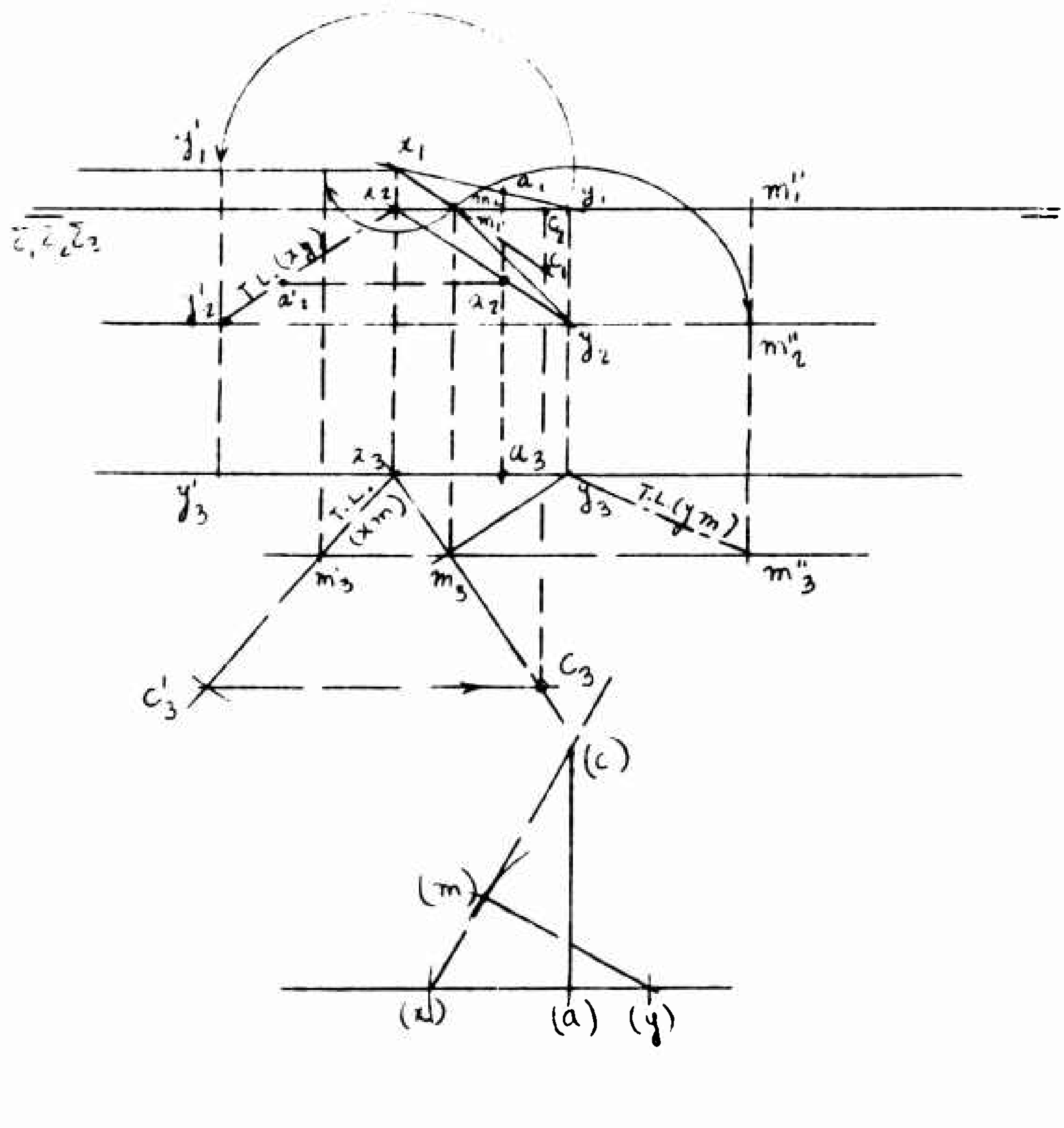


Figure 20

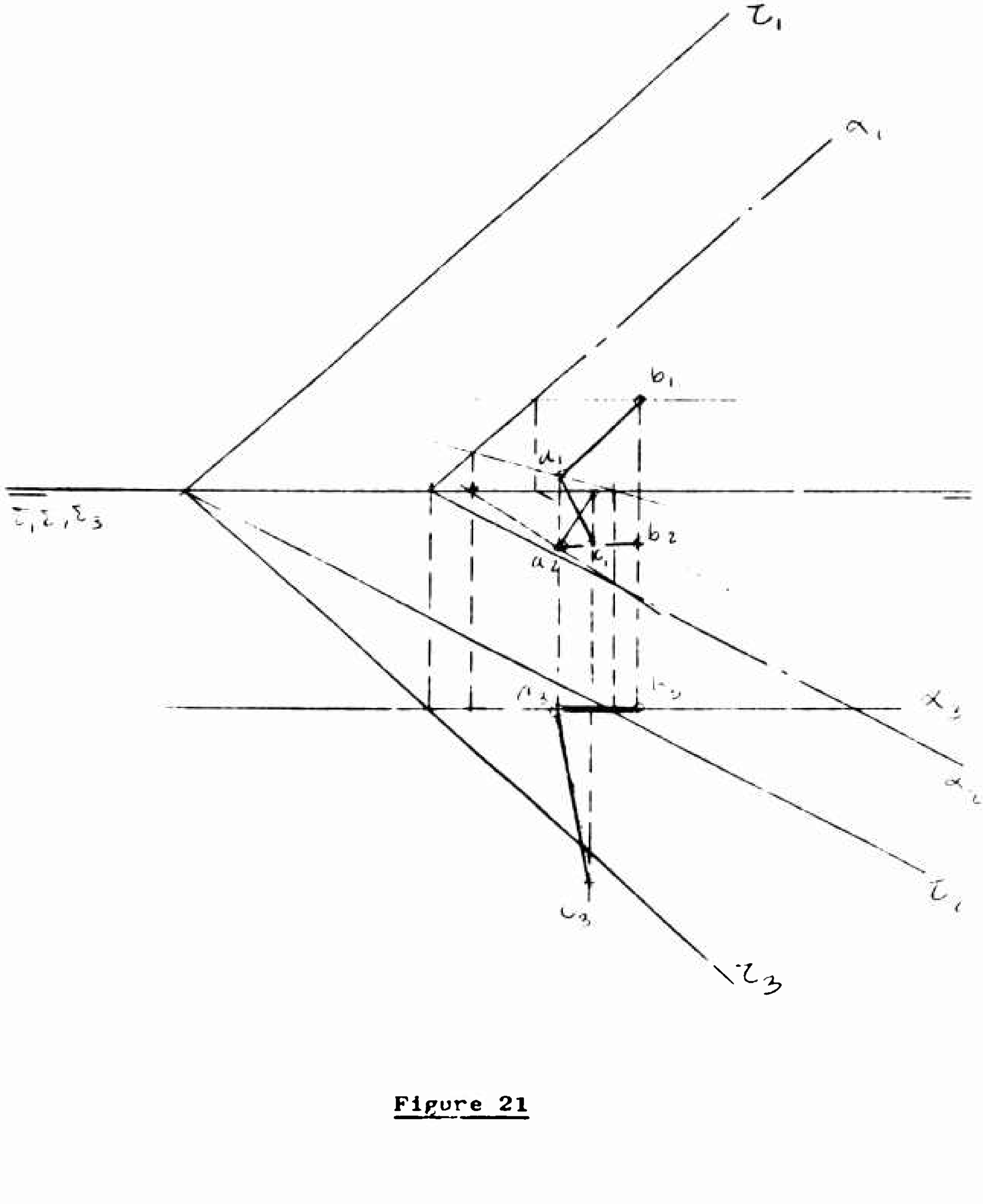


Figure 21

To determine (bc) perpendicular to (xy), in plane $\beta \equiv$ (mnp), apply changes in the 3-D spaces of reference, until β belongs to one of them, or determine the true lengths of the sides of the triangle (xym), draw (ac) perpendicular to (xy), and determine the projections of (c). See Figure 20.

Thus, we obtained the sides (ab) and (ac) of the plane angle that measures the angle of the two planes α and β (Figure 21).

Apply the solution of Problem No. 1 to obtain the true value of the angle.

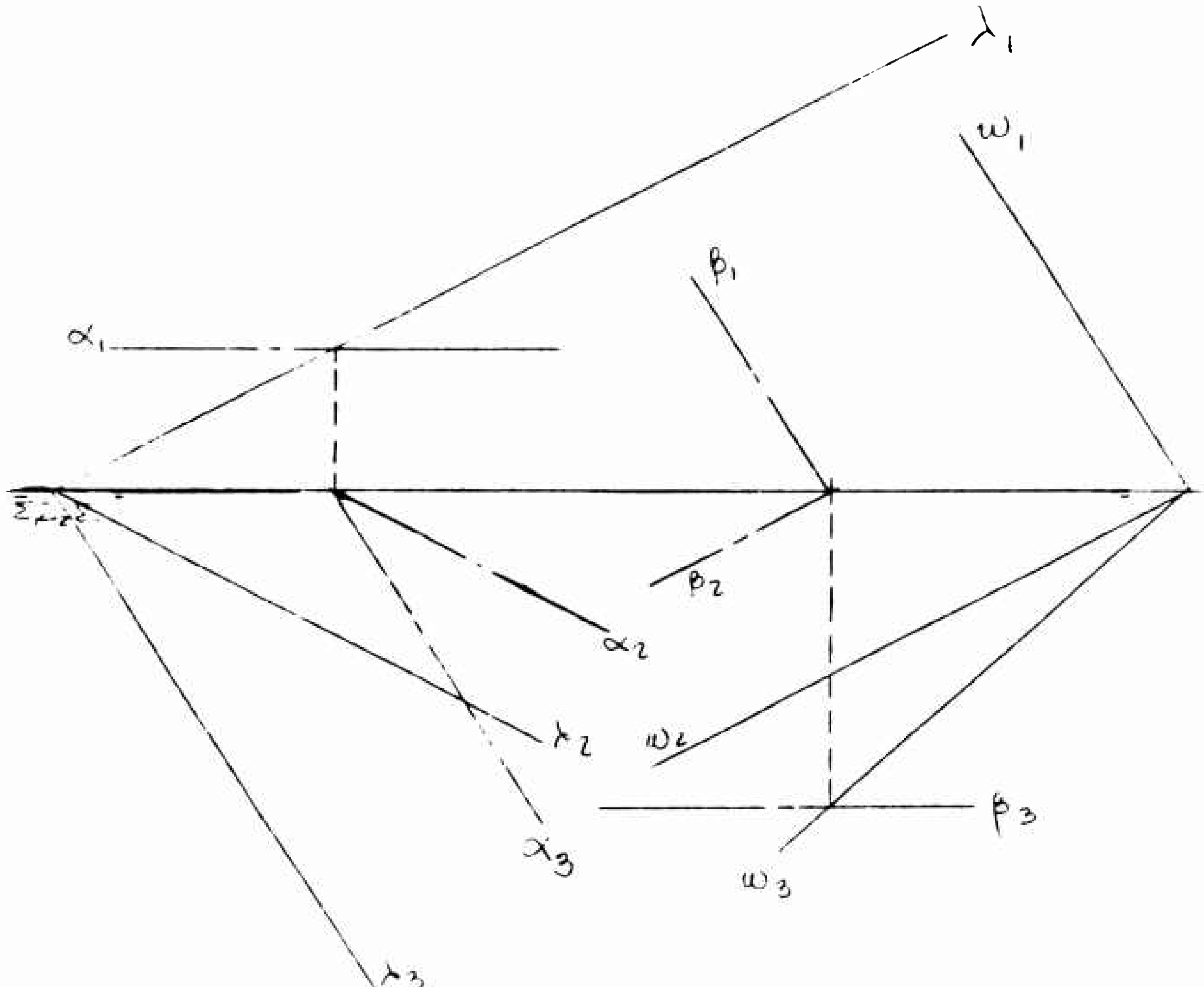
Problem No. 4

"To determine the true value of the angles of two planes belonging to distinct 3-D spaces".

Solution

Let Λ and Ω be two given 3-D spaces, and α a plane of Λ, and β a plane of Ω. (Figure 22).

First determine the plane of intersection of the two 3-D spaces. That is plane (abc) $\equiv \gamma$, Figure 23.



<u>**Figure 22**</u>

Next, determine the intersection of planes (abc) and α, in 3-D space Λ. That 's line (mn), Figure 24.

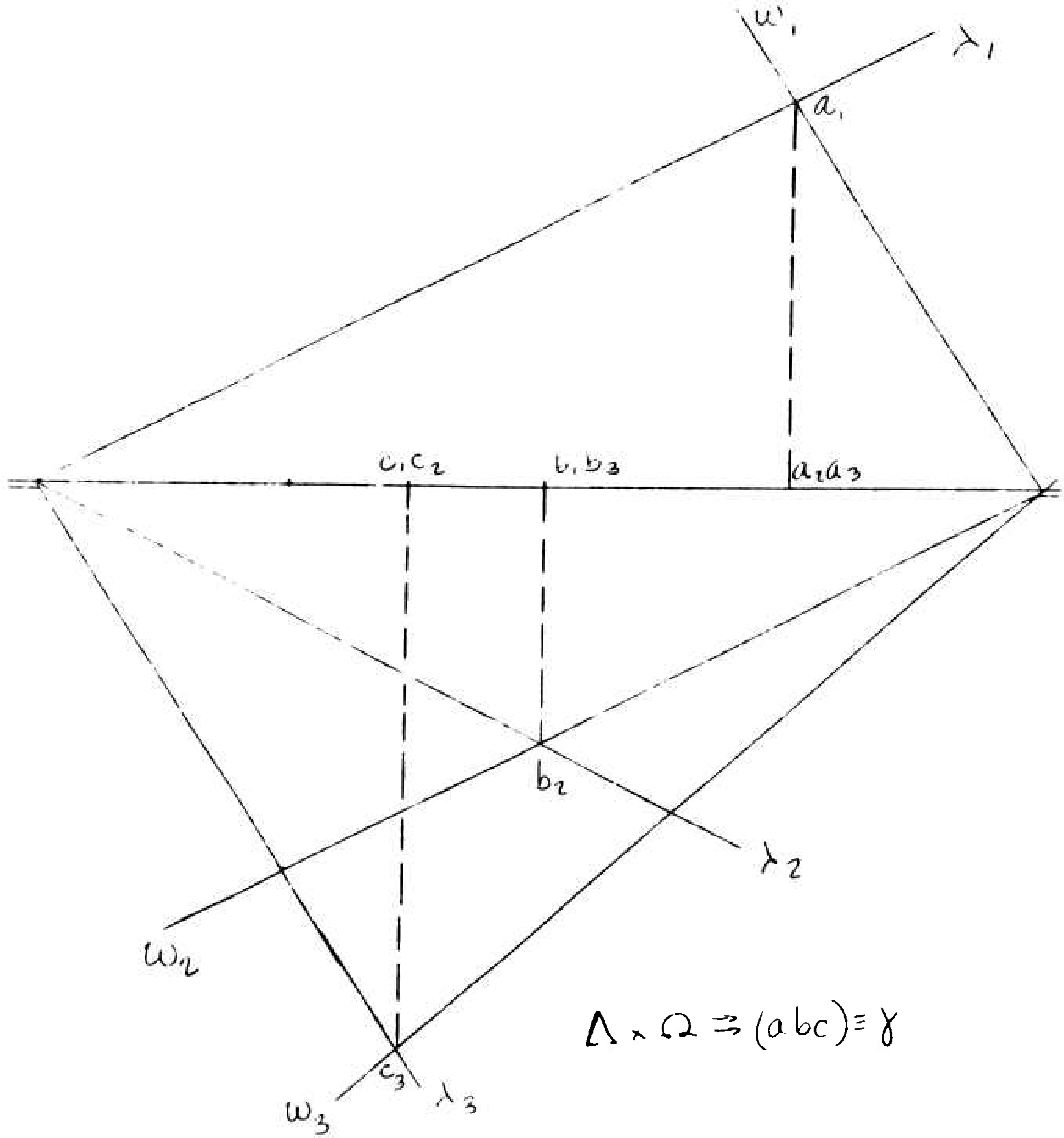
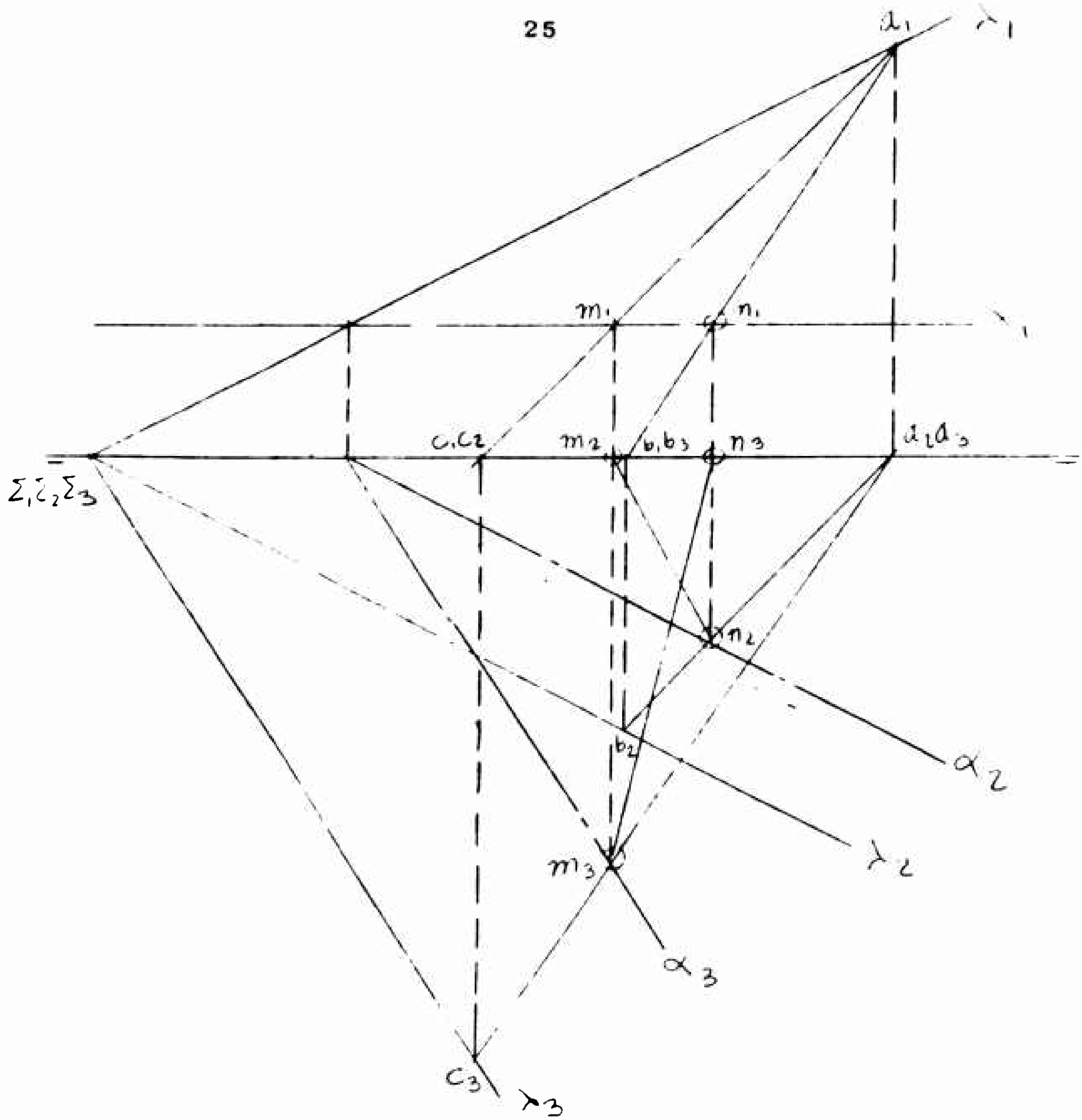


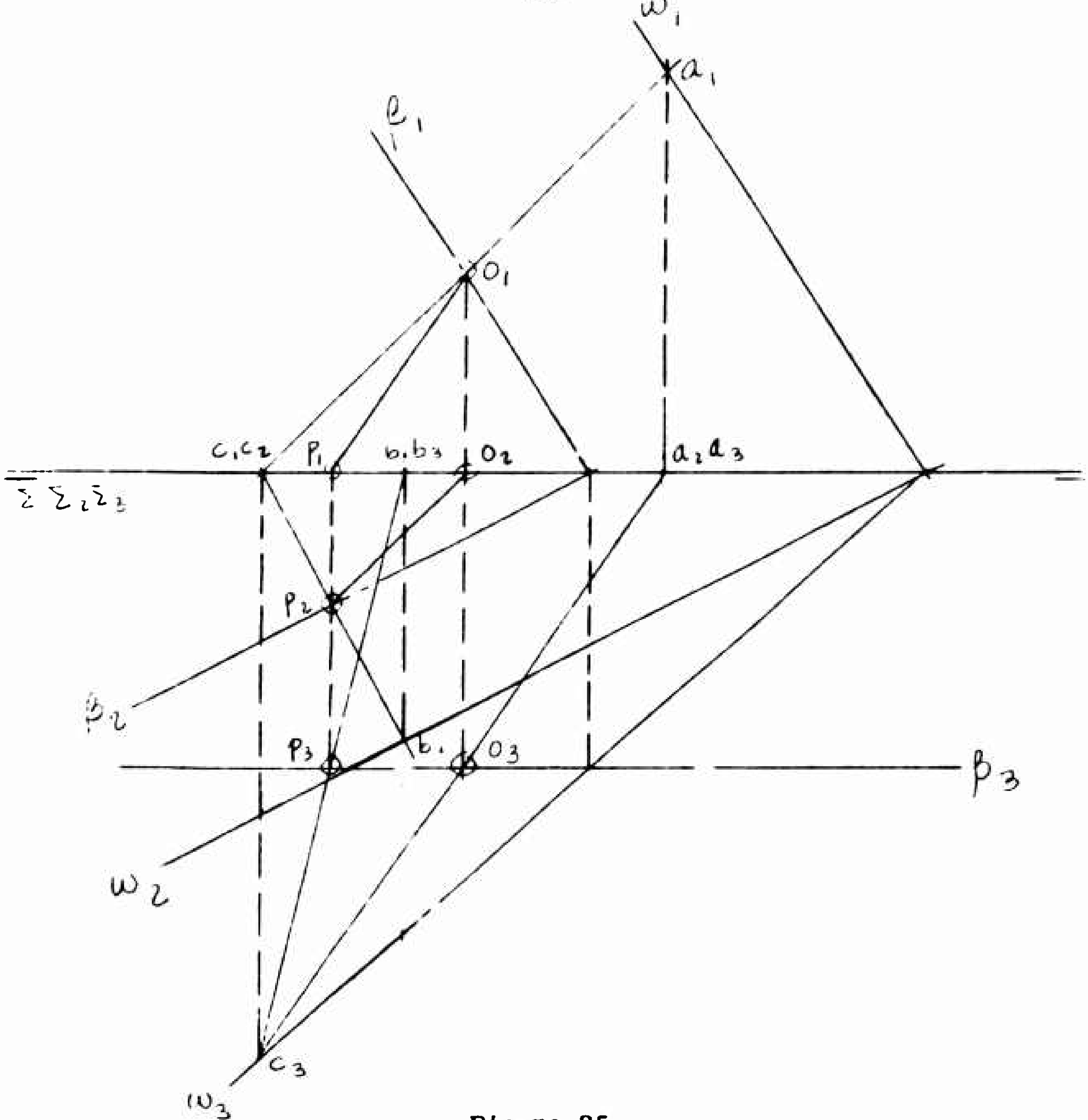
Figure 23

Next, determine the intersection of planes (abc) and β, in a 3-D space Ω. That, is line (op), Figure 25.



<u>**Figure 24**</u>

Finally, determine the intersection of lines (mn) and (op), point (s), in plane (abc). (Figure 26). The point (s) is the intersection of planes α and β .



<u>**Figure 25**</u>

Figure 27 shows all the constructions just outlined.

The point (s) is the vertice of the two minimal angles between the two planes.

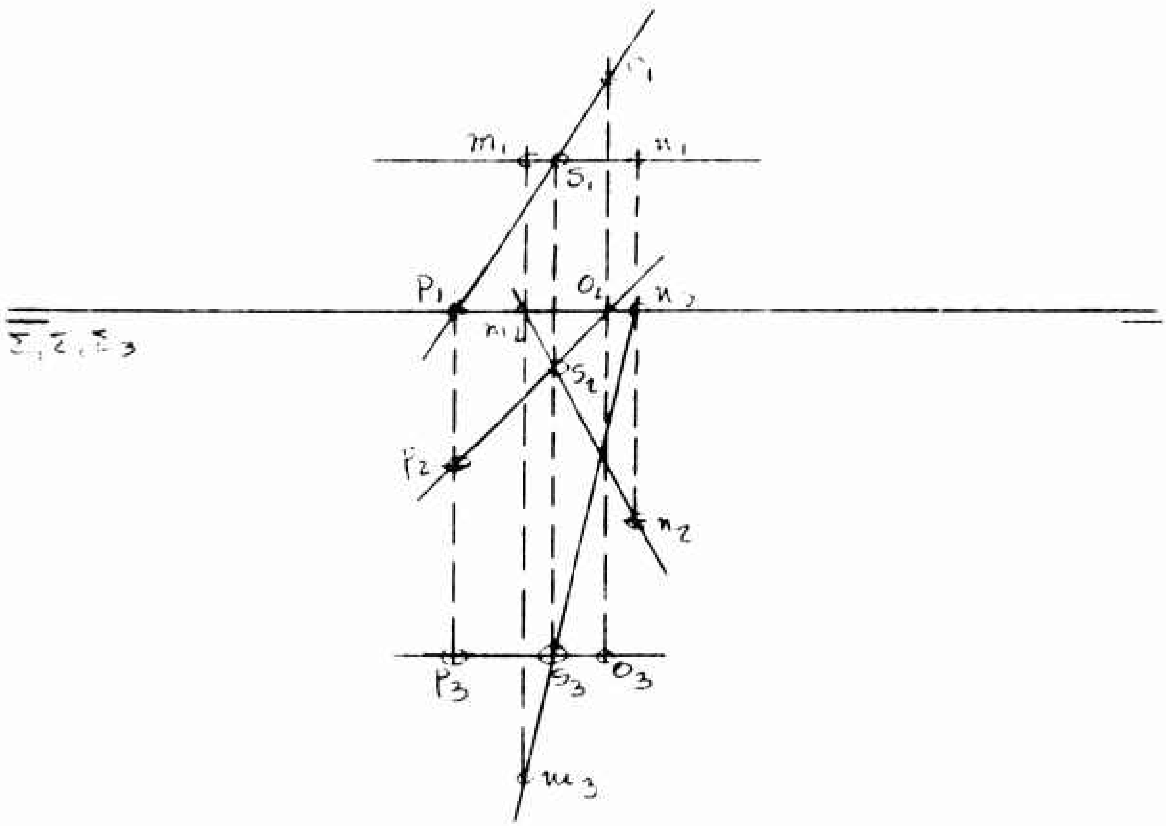
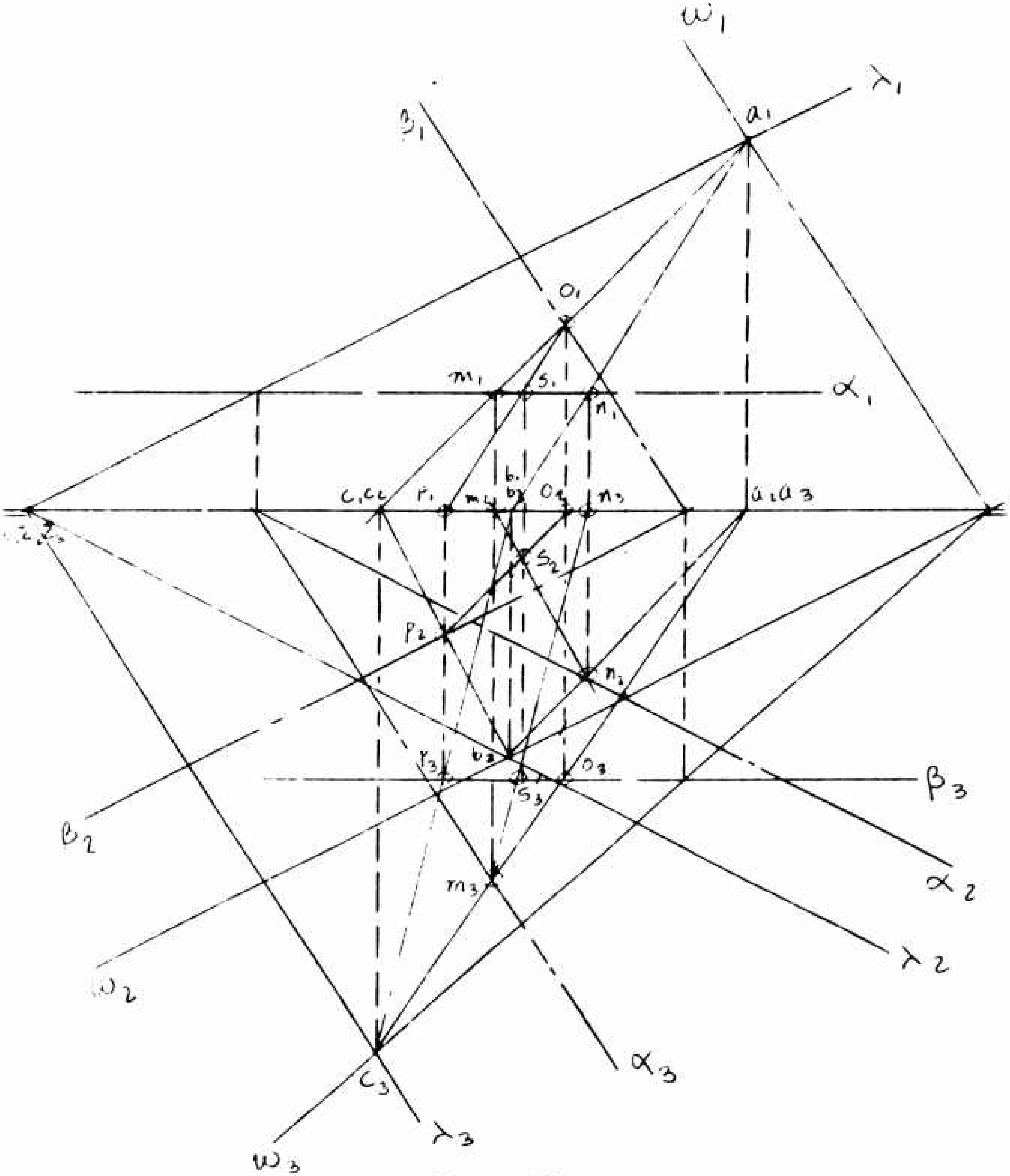


Figure 26

To determine the sides of the first angle, through (s) draw (st) perpendicular to plane α , in 3-D space Λ , and (sv) perpendicular to plane β , in 3-D space Ω . (Figure 28).



ω1
λ1
β1
a1
o1
m1
s1
n1
α1
p2
n2
r3
b3
o3
s3
β3
β2
α2
m3
λ2
ω2
α3
c3
λ3
ω3

Figure 27

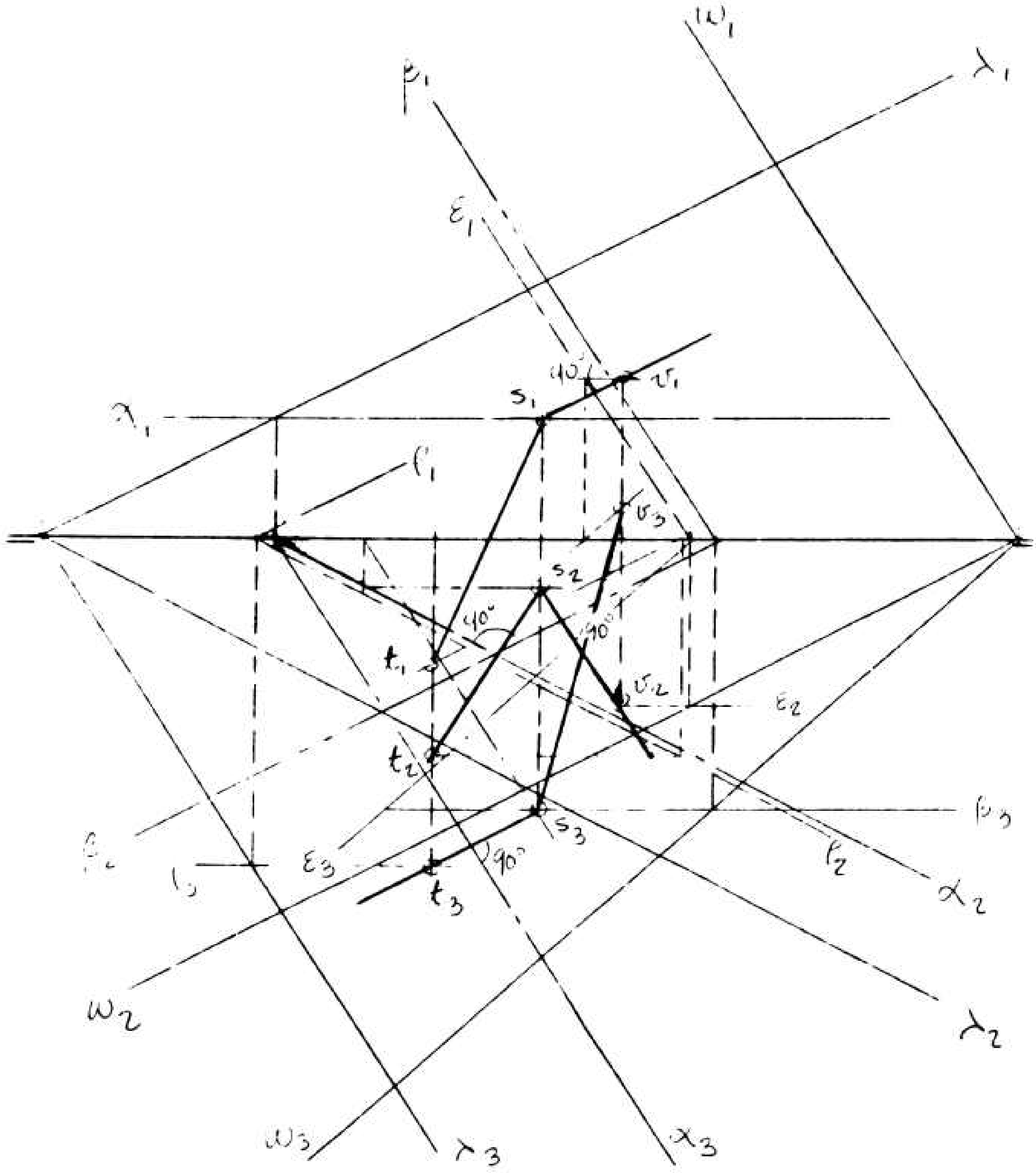


Figure 28

The first minimal angle is $\varphi = 180° - (t\hat{s}v)$

To determine the second angle, through (s) raise (sy) perpendicular to 3-D space Λ and (sx) perpendicular to 3-D space Ω. (Figure 24).

The second minimal angle is $\theta = 180° - (x\hat{s}y)$

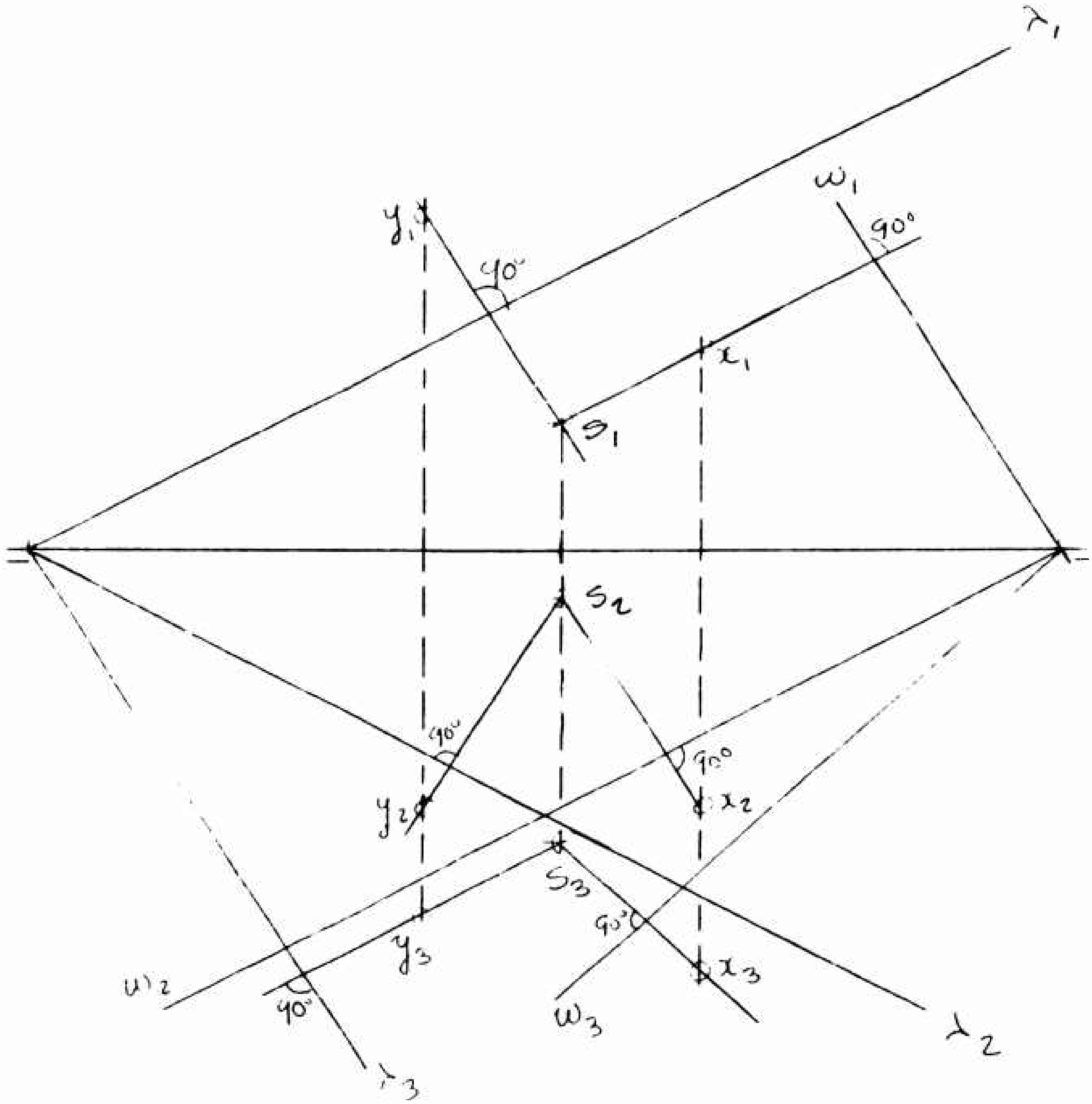

λ_1
ω_1
y_1
$90°$
$90°$
x_1
s_1
s_2
$90°$
$90°$
y_2
x_2
s_3
$90°$
y_3
x_3
ω_2
$90°$
ω_3
λ_2
λ_3

Figure 29

Problem No. 5

"To determine the true value of the angle between a line and a 3-D space".

Solution

The angle of a line and a 3-D space is equal to the angle between the line and its orthogonal projection on the 3-D space.

Given the line (ab) and the 3-D space Λ (Figure 30), through (a) and (b) draw perpendiculars to Λ . We have thus defined a plane α , perpendicular to Λ . (Figure 31).

To obtain the intersection of α and Λ (which will be the projection of (ab) on Λ), consider a 3-D space T belonging to α - Figure 32. The intersection of T and Λ is a plane β . (Figure 33). The intersection of α and β is the projection (cd) of (ab) on Λ . The two lines, (ab) and (cd) will be concurrent at a point (o), vertice of the angle. (Figures 34 and 35).

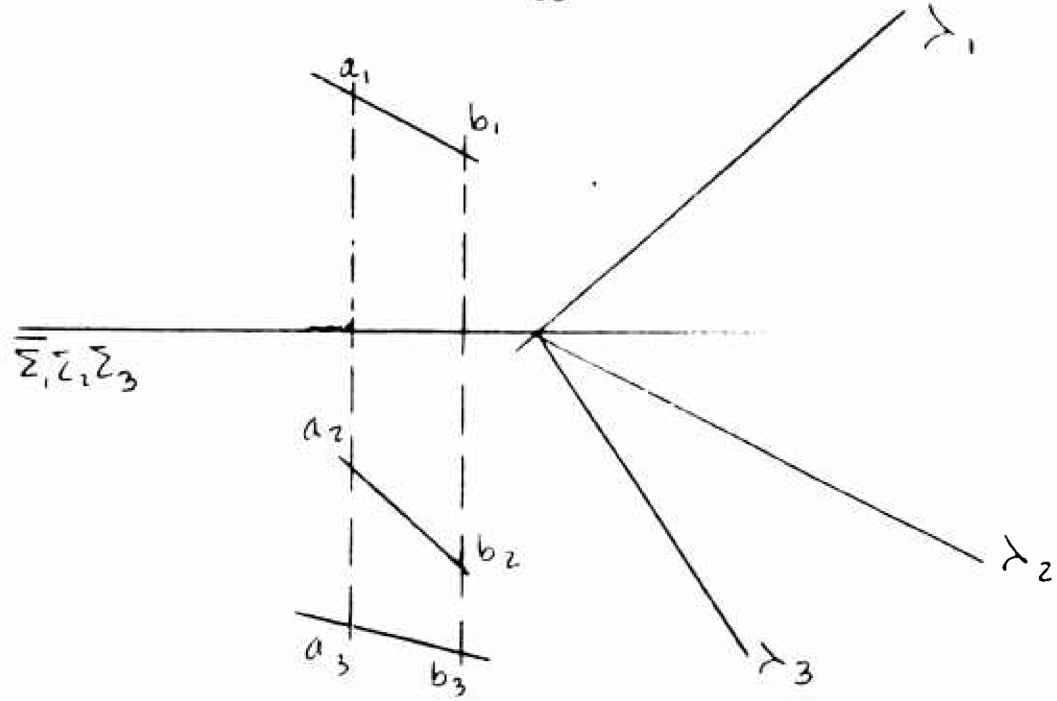


Figure 30

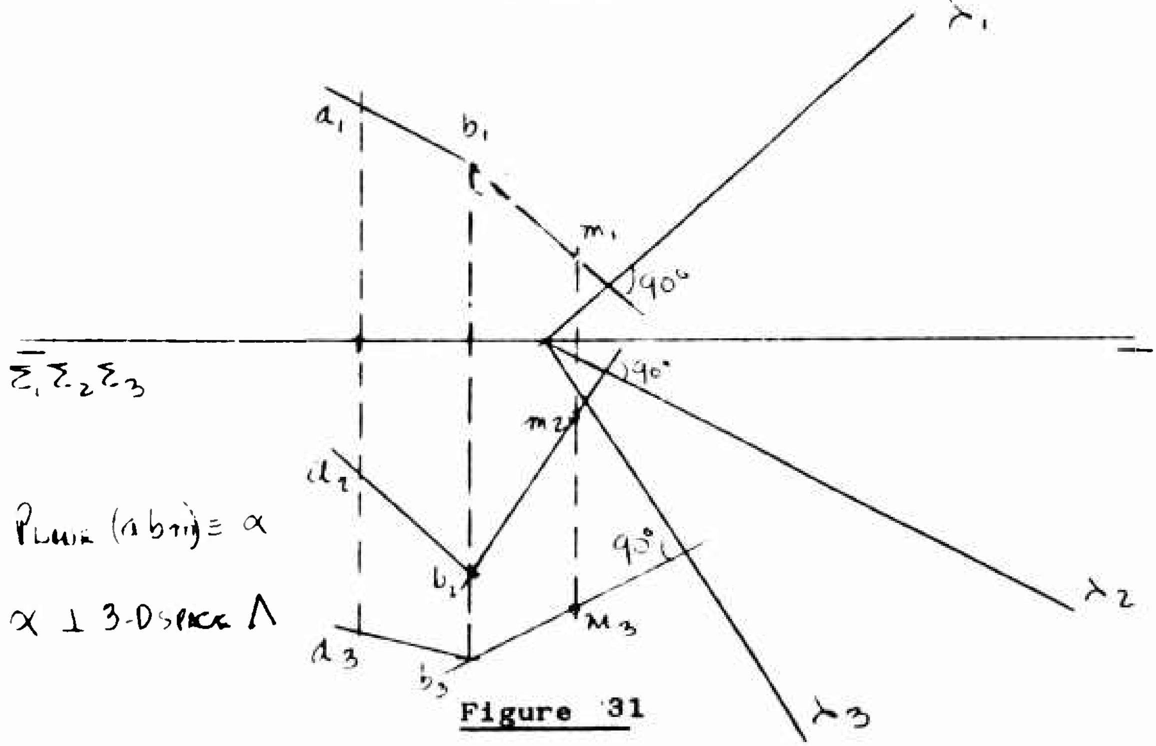


Figure 31

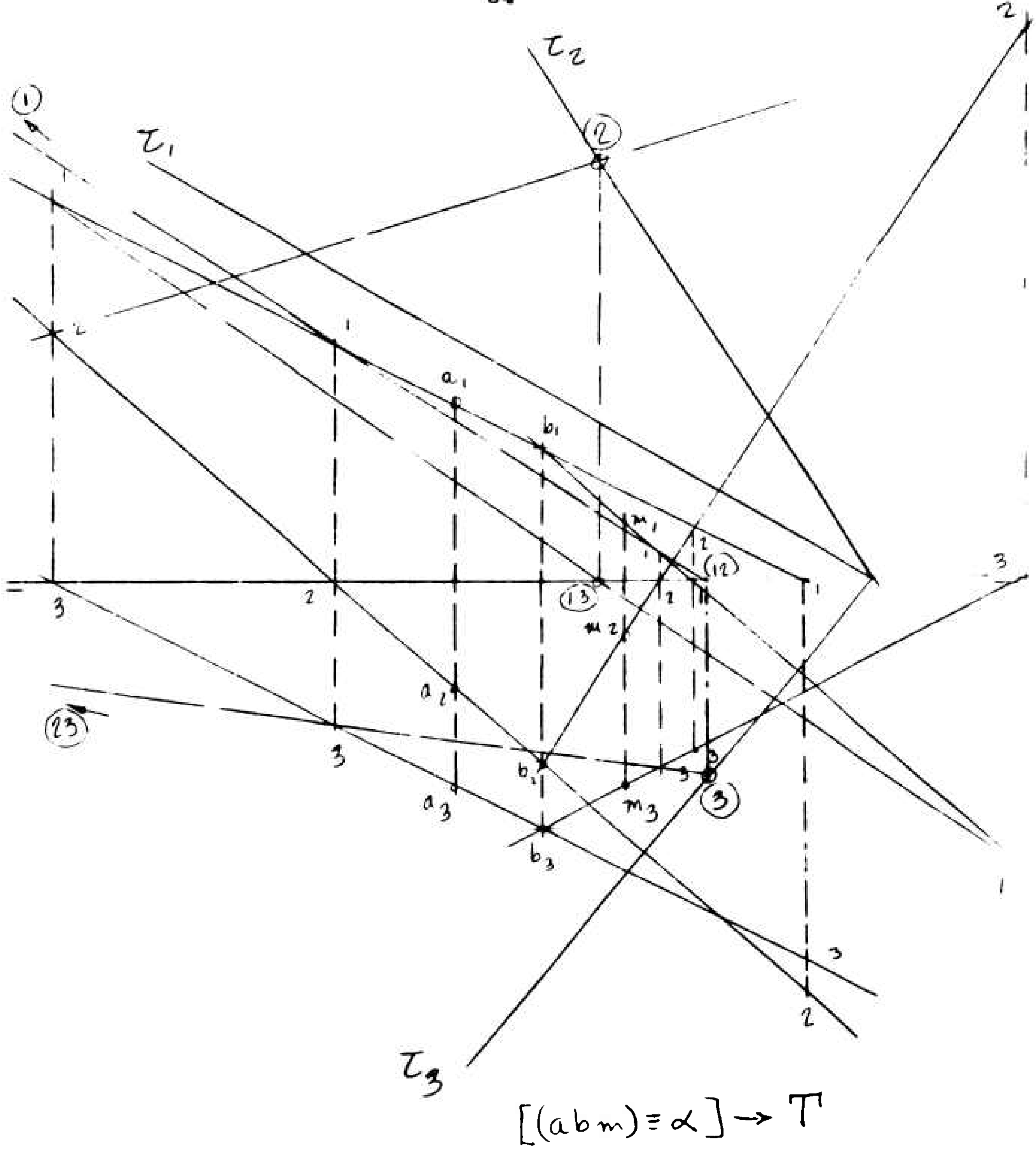

τ_1
τ_2
τ_3
a_1
a_2
a_3
b_1
b_2
b_3
m_1
m_2
m_3
(1)
(2)
(3)
(12)
(13)
(23)
$[(abm) \equiv \alpha] \to T$

Figure 32

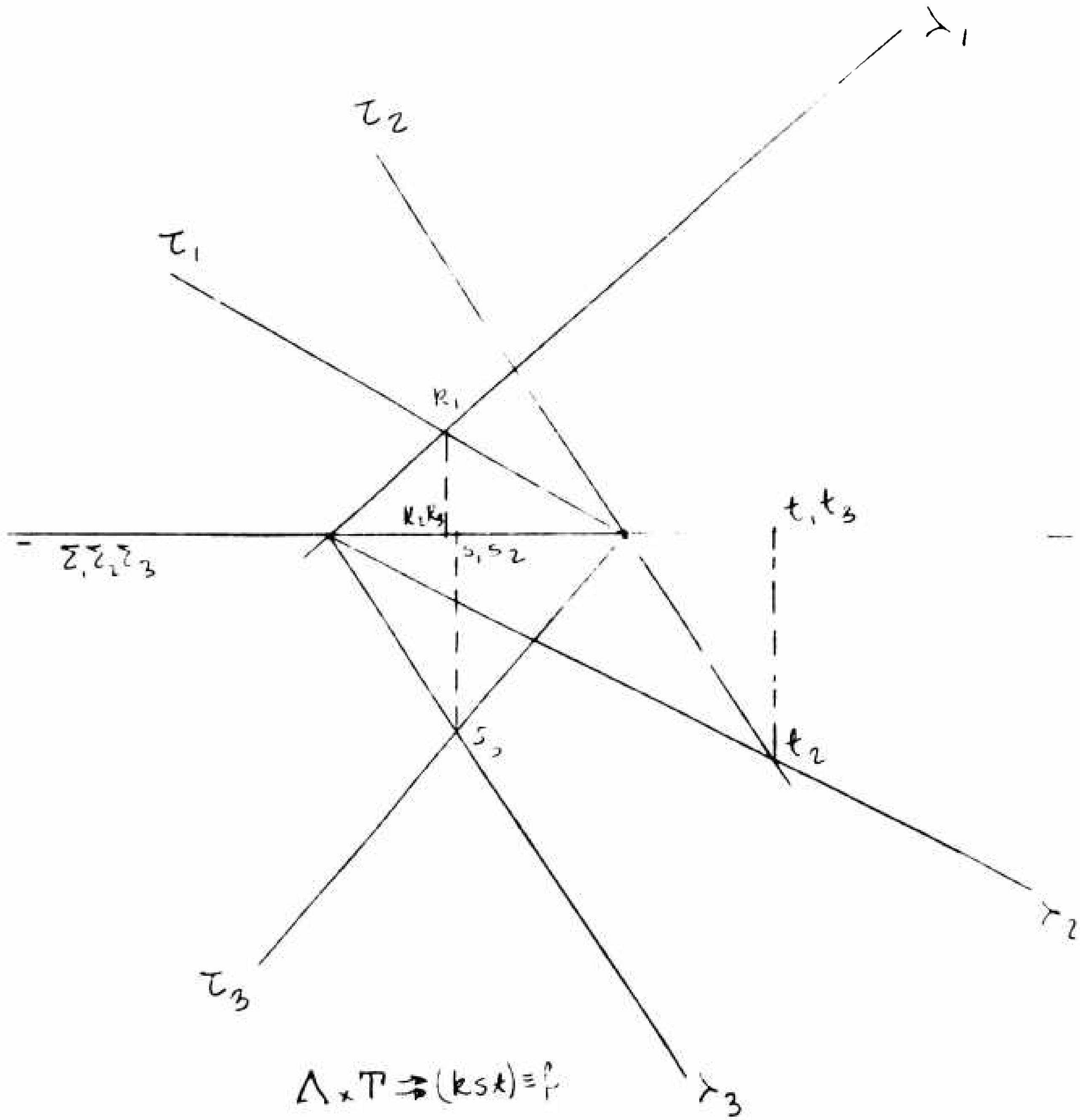


Figure 33

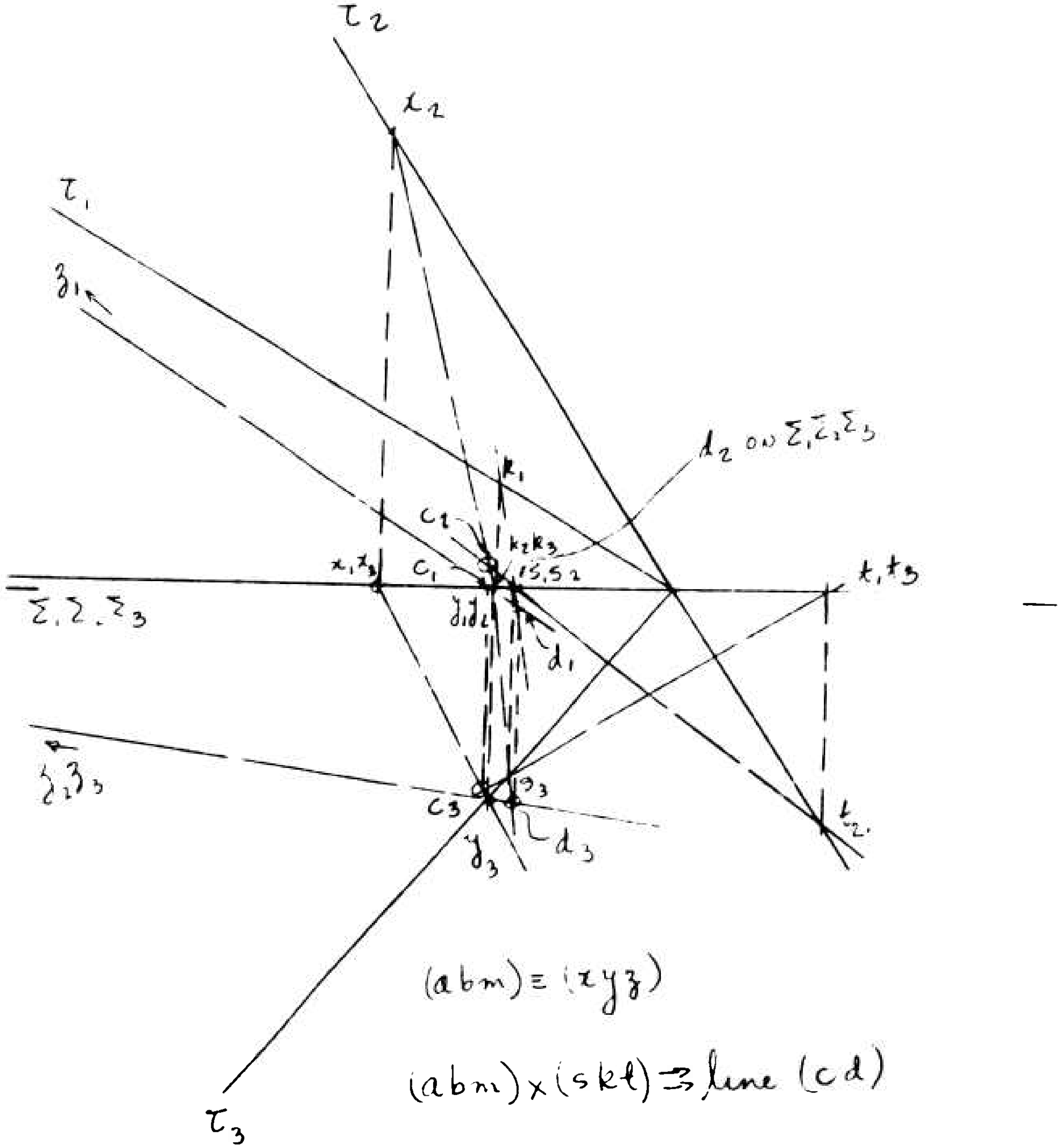


Figure 34

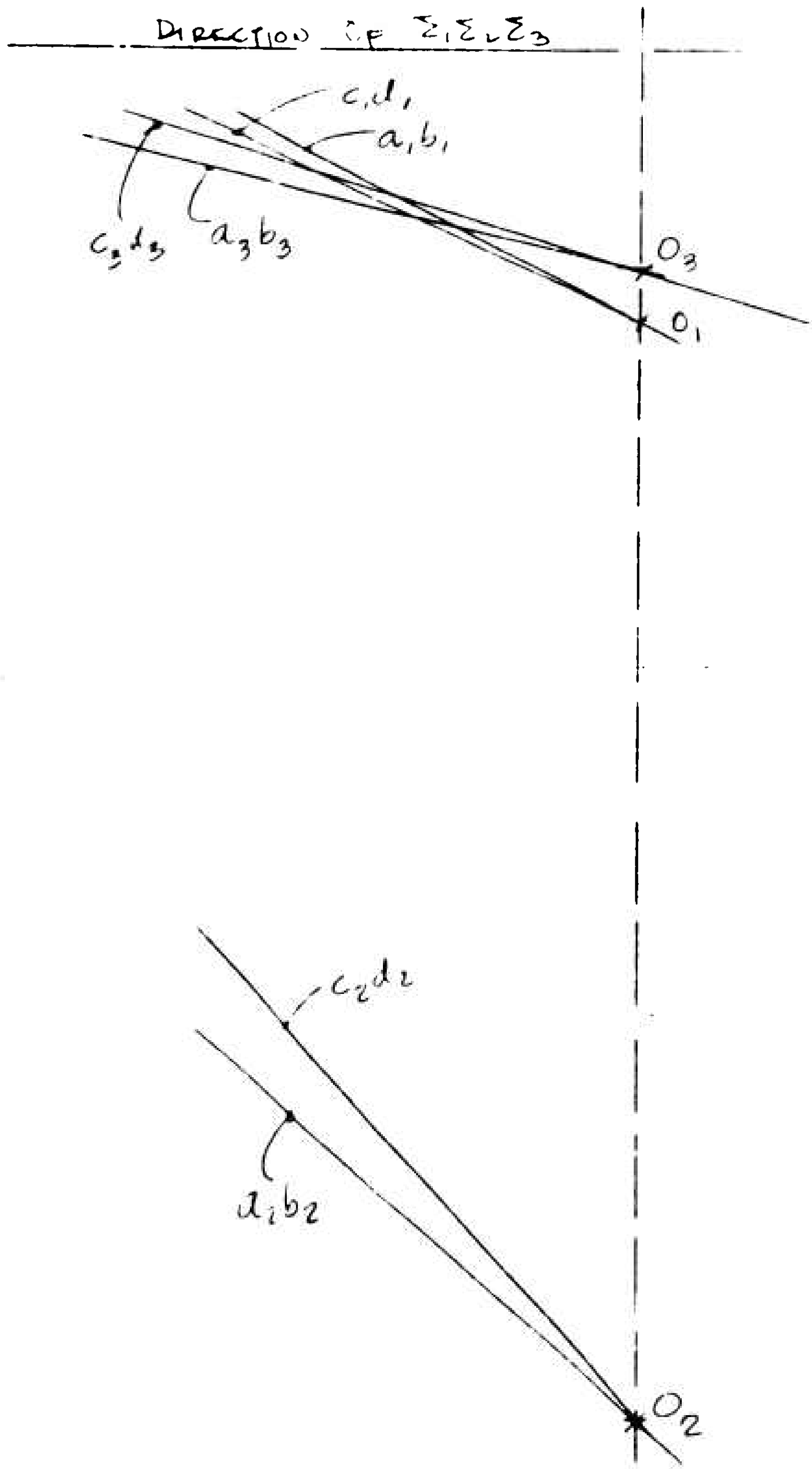

DIRECTION OF $\Sigma_1\Sigma_2\Sigma_3$
c_1d_1
a_1b_1
c_3d_3
a_3b_3
O_3
O_1
c_2d_2
a_2b_2
O_2

Figure 35

Problem No. 6

"To determine the angle of two 3-D spaces".

Solution

The plane angle which measures the angle of the two 3-D spaces has its vertice in the plane of intersection of the two 3-D spaces and its sides, one in each 3-D space, perpendicular to that plane of intersection.

Let Λ and Ω be the two given 3-D spaces. Their intersection is the plane (abc) $\equiv \alpha$, on which we mark a point (o), to be vertice of a plane angle. (Figure 36).

To determine the projections of (os) of Λ perpendicular to $\alpha \equiv$ (abc) and (ot) of Ω perpendicular to $\alpha \equiv$ (abc) it will be necessary to apply a descriptive method with one of the following objectives:

- Obtain superimposition of Λ and Ω (by independent application of the method) with one of the 3-D spaces of the 4-D system of reference;

- Apply the method so that one of the 3-D spaces becomes parallel to a 3-D space Σ_i of the 4-D system of reference.

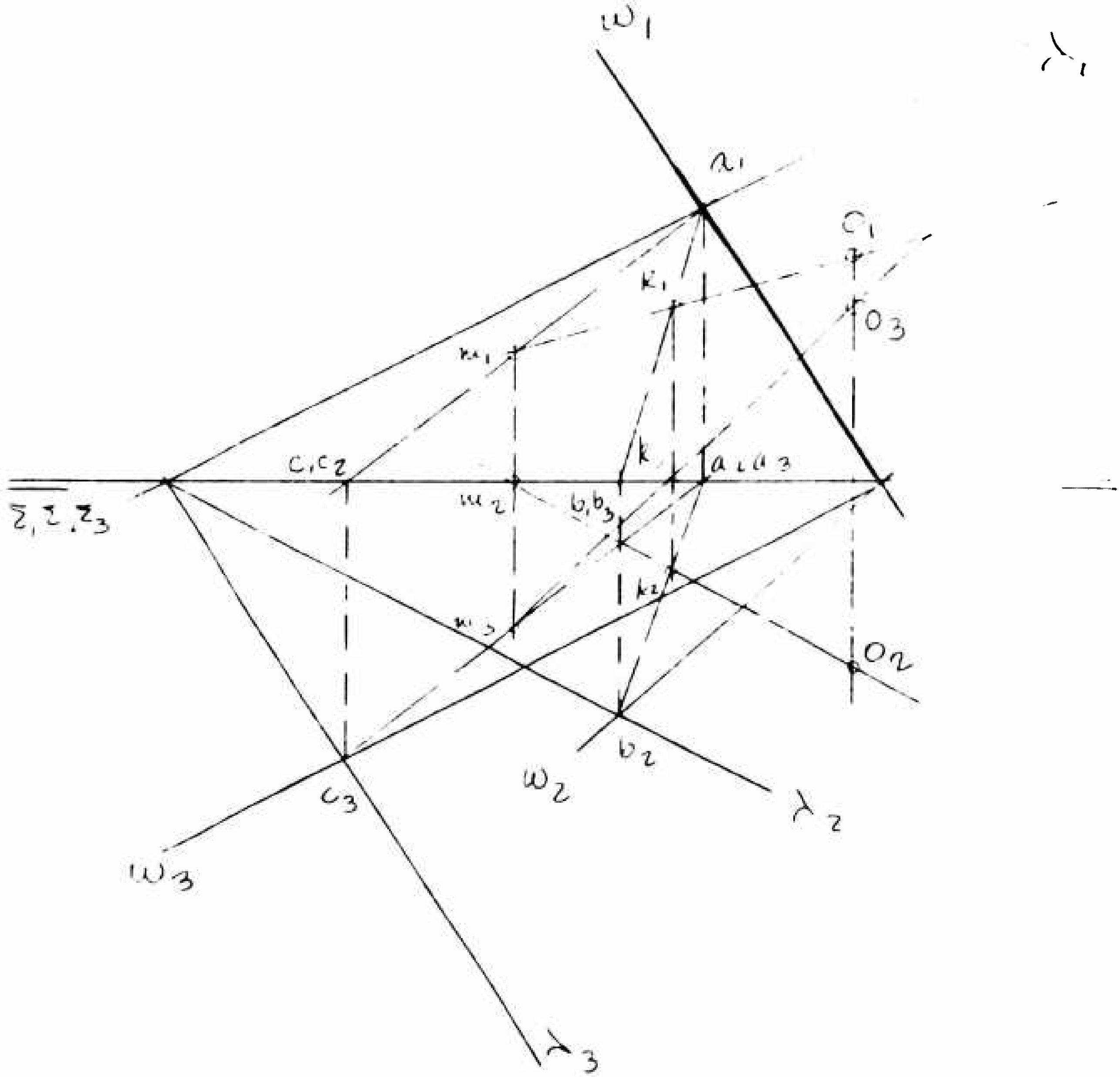


Figure 36

The result of the second alternative solution is shown in Figure 37. The 3-D space Ω is parallel to the 3-D space Σ_3.

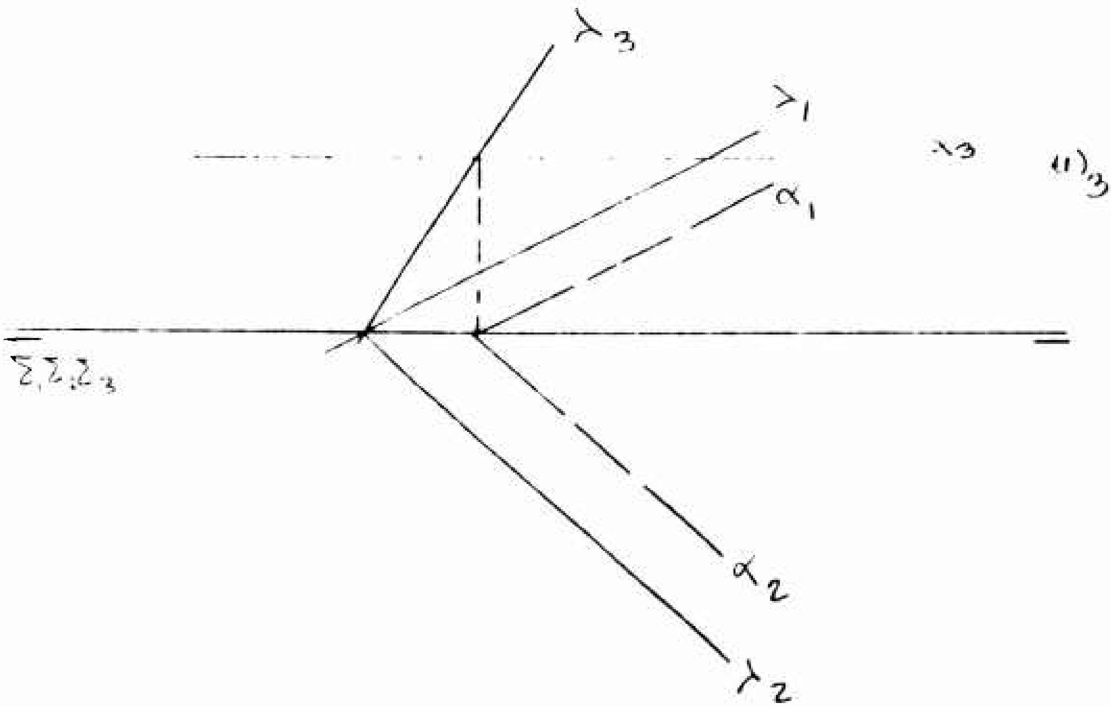
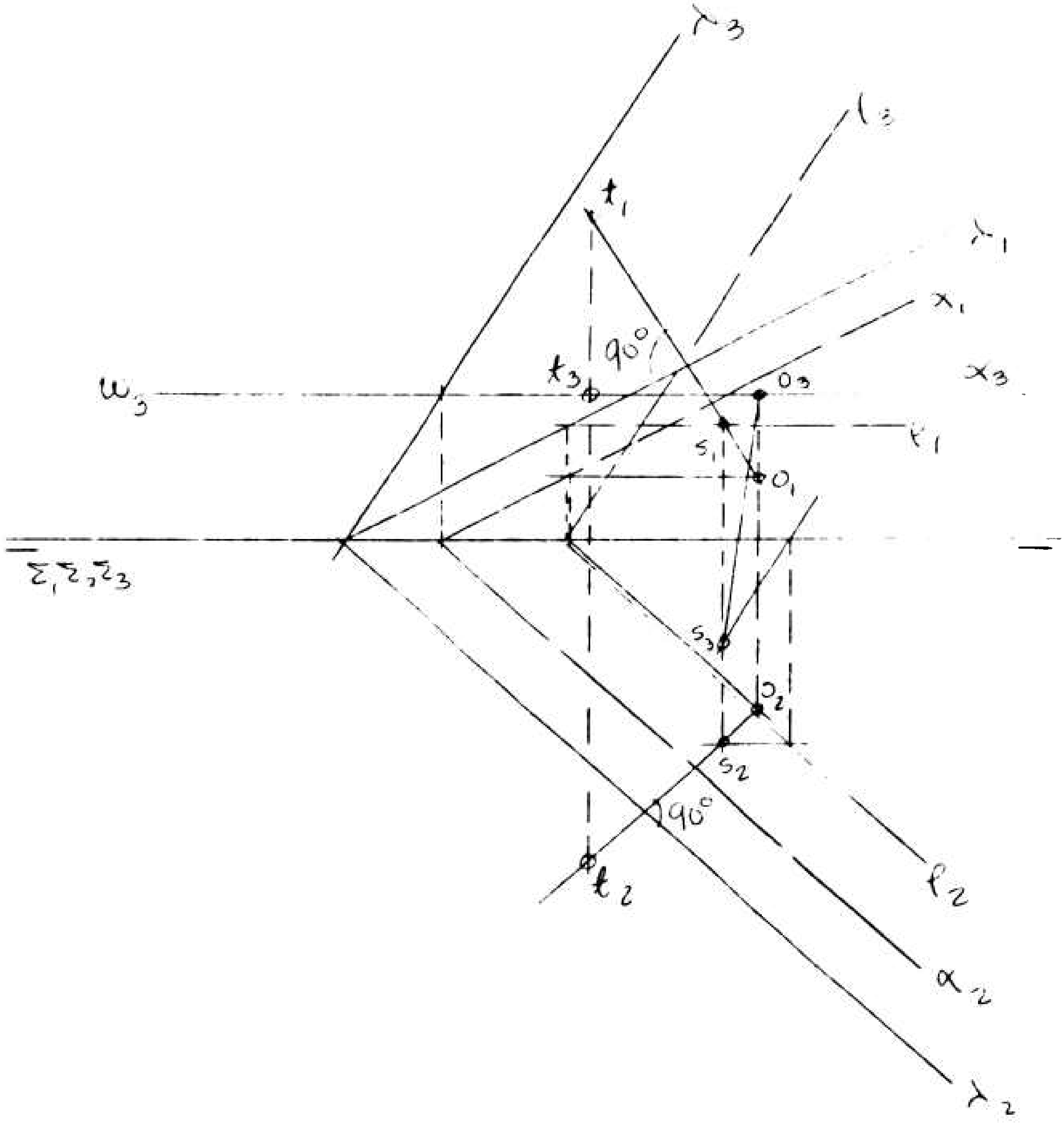


Figure 37.

In Figure 38 we show the perpendicular (os) to α, in the 3-D space Λ, and the perpendicular (ot) to α, in the 3-D space Ω. The true value of the angle $(t\hat{o}s) = \varphi$ may be obtained by applying the solution of Problem No. 1.



r_3
ℓ_3
t_1
r_1
x_1
$90°$
x_3
w_3
t_3
o_3
s_1
ℓ_1
o_1
$\Sigma_1\Sigma_2\Sigma_3$
s_3
o_2
s_2
$90°$
t_2
ℓ_2
α_2
λ_2

Figure 38

Problem No. 7

"To determine the true value of the angle between a plane and a 3-D space".

Solution

The angle that a plane makes with a 3-D space is the dihedral angle formed by the plane and the plane of intersection of a 3-D space belonging to the given plane and perpendicular to the given 3-D space.

Let (abc) be the given plane and $\mathcal{T}$ the given space. (Figure 39).

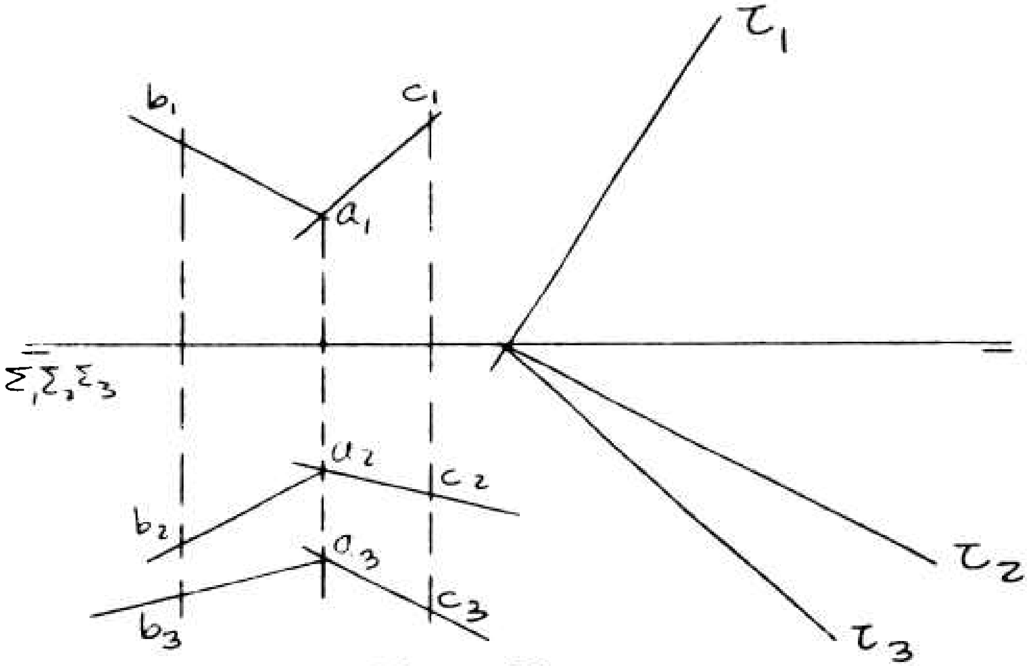
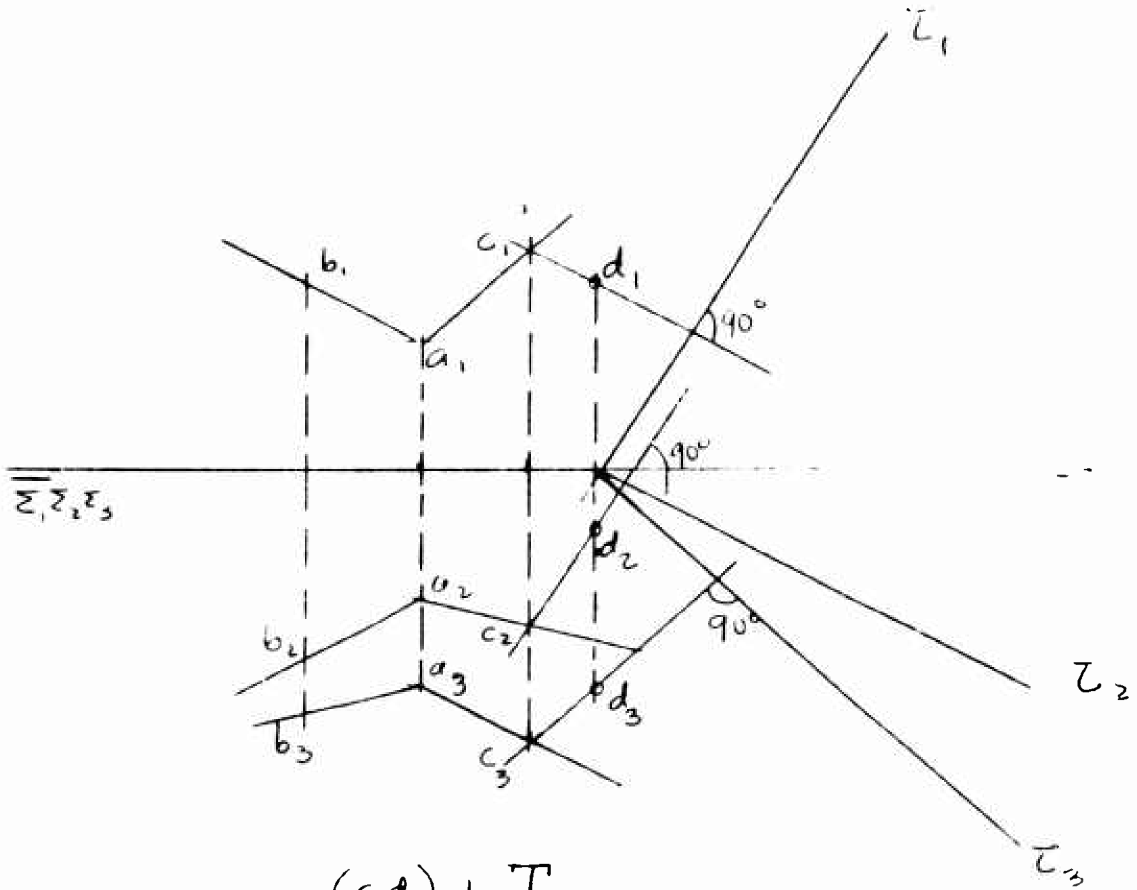


Figure 39

To define a 3-D space Ω perpendicular to T, and belonging to (abc), through a point of the plane draw a perpendicular to T. The plane (abc) and the perpendicular determine Ω. (Figure 40).



$$(cd) \perp T$$

$$\left[(abcd) \equiv \Omega\right] \perp T$$

Figure 40

To complete the problem, proceed as follows:

- Determine the intersection of Ω and T, plane β.

- Determine the angle between planes α and β, concurrent along a line, since both belong to the same 3-D space, Ω.